FIRST COURSE IN ALGEBRA

HAWKES-LUBY-TOUTON

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MATHEMATICAL TEXTS

FOR SCHOOLS

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FIRST COURSE IN ALGEBRA

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PREFACE

In this revision of their "First Course in Algebra" the authors have in general followed the plan of that text in the order of topics treated and in the method of their presentation.

The most important modification of the order of topics is found in the transference of the work on Ratio and Proportion to the last chapter in the book and the omission of the chapter on the Highest Common Factor and Lowest Common Multiple. The latter topic is treated in connection with the related material on fractions, while the former is placed among the Supplementary Topics at the end of the book.

Material for which there is no strong demand from teachers has been omitted, and the entire work has been rewritten in the interest of greater simplicity and directness of appeal. The collections of exercises and problems are for the most part new and contain a larger proportion of easy exercises with simple results than the first edition.

A striking feature of the revision is the inclusion of a large number of oral exercises in connection with the introduction of each new idea or operation. It is the object of these exercises to present the new concept in complete isolation from any complication of notation or technique so that the student becomes familiar with its content and bearing before he is asked to make use of it in written work. These oral exercises may well be taken up when the advance lesson is assigned, so that the pupil may be certain that he understands the idea involved in the new work before he leaves his instructor.

Another feature scarcely less important is the character and position of the examples and hints. The aim has been to

help the student at the exact point where he needs it and to avoid the insertion of lengthy and difficult solutions before they can be completely understood.

The definitions and axioms have been expressed in the simplest language which is consistent with scientific accuracy. Many definitions which are usually found in elementary texts but which do not contribute to the clearness of the subject are omitted.

The first presentation of the subject of graphs has been limited to the study of the straight line and a few exercises of a commercial or scientific character. These exercises not only have a very definite human interest apart from their mathematical value but also serve to familiarize the student with the kind of graphs he will meet in his ordinary reading.

The first consideration in the treatment of radicals has been the needs of the student for his later study of the quadratic equation and for his work in geometry.

Frequently a student's knowledge of algebra is limited to a greater or less facility in the use of the rules of operation — to mere technique. To obviate this result the development of the problem work in this text has received full and careful attention.

The authors have received suggestions of great value from many teachers in all parts of the country, for which they extend their thanks. They are under especial obligation to Mr. E. L. Brown, of Denver, Colorado, Professor H. E. Cobb, of Chicago, Illinois, and to Mr. L. A. Pultz, of Rochester, New York, for helpful criticism.

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FIRST COURSE IN ALGEBRA

CHAPTER I

INTRODUCTION

1. The numbers and symbols of arithmetic. The simple operation of counting employs the numbers we call integers. To represent these integers and the other numbers with which it deals, arithmetic uses the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Operations on the numbers of arithmetic are indicated by the symbols $+$, $-$, $\times$, and $\div$. The operation of division applied to integers gives rise to fractions. With these two kinds of numbers, integers and fractions, the student's work in arithmetic is mainly carried on.

2. Symbols of algebra. Symbols are employed far more extensively in algebra than in arithmetic, and many new ideas arise in connection with their meaning and their use. Some symbols represent numbers, others indicate operations upon them, others represent relations between them, and still others represent kinds of numbers with which arithmetic does not deal. Letters as well as arabic numerals are used to represent numbers. The following symbols of operation, $+$, $-$, $\times$, and $\div$, have the same meaning as in arithmetic. The sign of multiplication is usually replaced by a dot or omitted.

For example, $3 \times a$ is written $3 \cdot a$, or $3a$, and $2 \times a \times b$ is written $2ab$. Also $a \div b$ is often written $\frac{a}{b}$.

The sign = is read *equals*, or *is equal to.* As the need for them arises, other symbols will be introduced.

3. The use of letters to represent numbers. The use of the letters of the alphabet to represent numbers is the most striking difference between arithmetic and algebra.

In arithmetic we speak thus: If the sides of a triangle are 6, 7, and 9 inches respectively, its perimeter is $6+7+9$, or 22 inches. The corresponding statement in algebra is: If the sides of a triangle are a, b, and c inches respectively, and its perimeter is p inches, then $p=a+b+c$. Here the second statement is true for every triangle, while the first is not true for every triangle.

Similarly: If a rectangle is 8 inches by 12 inches, its perimeter is $8+12+8+12$, or 40 inches. And if a rectangle is l inches long and w inches wide and if p denotes its perimeter in inches, then $p=l+w+l+w$, or $2l+2w$. Here, again, the arithmetical statement is particular and applies to one rectangle only, while the algebraic statement is general; that is, it is true for all rectangles.

The gain in power which the general symbolic language of algebra affords over the particular numerical language of arithmetic constitutes one of the most important advantages of the algebraic method. As the student progresses he will meet with many illustrations of this feature of algebra.

The purpose of the following exercises is to familiarize the student with the use of letters in the place of numbers.

ORAL EXERCISES

1. What numerical value has $5a$ when a is 3? when a is 5? when a is 10?

2. What numerical value has $6a+2b$ when a is 2 and b is 4?

3. Express $h+3m$ in seconds if h and m stand for the number of seconds in an hour and in a minute respectively.

4. Express $10y + 4f$ in inches if y and f stand for the number of inches in a yard and in a foot respectively.

5. If q and d represent the number of cents in a quarter and in a dime respectively, express $4q + 6d$ in cents.

6. If t and h represent the number of pounds in one ton and in one hundredweight respectively, express $4t + 6h$ in pounds.

7. $3x + 5x =$ how many x?

8. $4x + 5x = (?)x$.

9. $5 \cdot t + 10 \cdot t = (?)t$.

10. $5x + 3x + 6x = ?$

11. $3x - 2x + 7x - 5x = ?$

12. 7 books + 3 chairs + 2 books + 5 chairs = (?) books + (?) chairs.

13. 8 books + 4 chairs − 2 chairs + 4 books = (?) books + (?) chairs.

14. 6 books + 7 chairs − 3 books − 2 chairs = (?) books + (?) chairs.

15. $5b + 4c + 8b - 2c = (?)b + (?)c$.

16. $6b + 3x + 4b + 8x = (?)b + (?)x$.

17. $6x + 2b + 3x - b + x = (?)b + (?)x$.

18. $2x + 2 + 3x + 4 = (?)x + ?$

19. $4x + 2 + 3x + 2 - x + 8 = ?$

20. $x + x + 2 + x + x + 2 = ?$

21. $n + n + 1 + n + 2 = ?$

22. $5a + 18 - 3a - 7 = ?$

23. $8x - 3 + 18 - 5x = ?$

24. $4w - 8 + 3w + 20 = ?$

25. What value has $4x + 3$ when $x = 2$? when $x = 7$?

26. What value has $3x - 4$ when $x = 3$? when $x = 2$?

27. The side of a square is 5 inches long. What is its area? its perimeter?

28. The side of a square is s inches long. What represents its perimeter? its area?

29. The base of a rectangle is 12 feet, and its altitude is 4 feet. What is its perimeter? its area?

30. If b represents the number of feet in the base of a rectangle and a the number of feet in its altitude, what is its perimeter? its area?

31. A rectangle is twice as long as it is wide. Let w represent the number of inches in its width. Then express, in terms of w, (*a*) the length; (*b*) the perimeter; (*c*) the area.

32. A man is three times as old as his son. If s denotes the number of years in the son's age, what will represent the father's age?

33. A father is 28 years older than his son. If s represents the son's age in years, what will represent the father's age?

34. A rectangle is 24 inches longer than it is wide. Let b represent the width in feet. Then represent the length and the perimeter in terms of b and numbers.

35. A rectangle is 16 feet narrower than it is long. If w represents the width in feet, what will conveniently represent the length? the perimeter?

36. A rectangle is 4 feet longer than twice its width. Express the width, the length, and the perimeter in terms of a letter, or a letter and numbers.

Origin of symbols. Many of the symbols that are in common use in algebra at the present time have histories which not only are interesting in themselves, but which also serve to indicate the slow and uncertain development of the subject. It is often found that symbols which seem without meaning represent some abbreviation or suggestion long since forgotten, and that operations and methods which we find hard to master have sometimes required hundreds of years to perfect.

In the early centuries there were practically no algebraic symbols in common use; one wrote out in full the words *plus*, *minus*, *equals*, and the like. But in the sixteenth century several Italian mathematicians used the initial letters $\bar{p}$ and $\bar{m}$ for $+$ and $-$. Some think that our modern symbol $-$ came into use through writing the initial m so rapidly that the curves of the letter gradually flattened out, leaving finally a straight line. The symbol $+$ may have originated similarly in the rapid writing of the letter p. But in the opinion of others these symbols were first used in the German warehouses of the fifteenth century to mark the weights of boxes of goods. If a lot of boxes, each supposed to weigh 100 pounds, came to the warehouse, the weight would be checked, and if a certain box exceeded the standard weight by 5 pounds, it was marked $100 + 5$; if it lacked 5 pounds, it was marked $100 - 5$. Though the first book to use these symbols was published in 1489, it was not until about 1630 that they could be said to be in common use.

Both of the symbols for multiplication given in the text were first used about 1630. The cross was used by two Englishmen, Oughtred and Harriot, and was probably an adaptation of the letter x, which is found some years earlier. The dot is first found in the writings of the Frenchman Descartes. It is interesting to note that Harriot was sent to America in 1585 by Sir Walter Raleigh, and returned to England with a report of observations. He made the first survey of Virginia and North Carolina, and constructed maps of those regions.

It is strange that the line was used to denote division long before any of the other symbols here mentioned were in use. This is, in fact, one of the oldest signs of operation that we have. The Arabs, as early as 1000 A.D., used both $\frac{a}{b}$ and a/b to denote the quotient of a and b. The symbol $\div$ did not occur until about 1630.

Equality has been denoted in a variety of ways. The word *equals* was usually written out in full until about the year 1600, though the two sides of an equation were written one over the other by the Hindus as early as the twelfth century. The modern sign $=$ was probably introduced by the Englishman Recorde, in 1557, because, he says, "Noe. 2. thynges can be moare equalle" than two parallel lines. This symbol was not generally accepted at first, and in its place the symbols $\parallel$, $\propto$, and ∞ are frequently met during the next fifty years.

4. The usefulness of symbols. Symbols enable one to abbreviate ordinary language in the solution of problems.

For example: Three times a certain number is equal to 20 diminished by 5. What is the number?

If n represents the number, the preceding statement and question can be written in symbols, thus:

$$3n = 20 - 5.$$
$$n = ?$$

The symbolic statement $3n = 20 - 5$ is called an *equation* and n the *unknown number*.

If $\quad 3n = 20 - 5,$

then $\quad 3n = 15,$

and $\quad n = 5.$

While the preceding example is very simple, it illustrates the algebraic method of stating and solving the problem. The method is brief and direct, and its advantages will become more apparent as the student progresses.

ORAL EXERCISES

Find the numerical value of x in the following equations:

1. $3x = 18.$

2. $4x = 28.$

3. $5x = 30.$

4. $7x = 42.$

5. $4x = 12 + 8.$

6. $3x = 4 + 11.$

7. $6x = 17 + 13.$

8. $4x + 3x = 35.$

9. $6x + 2x = 32.$

10. $5x + 4x = 45.$

11. $4x + 3x = 56.$

12. $7x + 2x = 15 + 3.$

13. $9x - 3x = 18 + 12.$

14. $4x - x = 15 - 6.$

15. $5x + 4x - 2x = 10 + 4.$

16. $4x + 3x - x = 33 - 3.$

17. $6x - x + 3x = 42 + 6.$

18. If one number is represented by x, what will represent a number three times as great?

19. James had $3x$ cents. His brother had four times as many. Represent the number of cents the brother had.

20. Paul's weight is $2x$ pounds, and his father weighs three times as much. What will represent the father's weight? the weight of the two together?

21. The area of a circle is $6y$. Represent the area of a circle three times as large.

22. One number is twice a second, and the second is four times the third. If x represents the third, what will represent the second? the first?

23. One newsboy has three times as many papers as a second, and the two together have as many as a third. Represent in terms of x the number of papers each has.

EXAMPLE

The sum of two numbers is 112. The greater is three times the less. What are the numbers?

Solution. By the conditions of the problem,

$$\text{greater number} + \text{less number} = 112. \qquad (1)$$

Let $\qquad l = $ the less number.

Then $\qquad 3l = $ the greater number.

Substituting these symbols in (1), we have

$$3l + l = 112.$$

Collecting, $\qquad 4l = 112.$

Whence $\qquad l = \frac{112}{4} = 28,$

and $\qquad 3l = 3 \times 28 = 84.$

Therefore the greater number is 84 and the less 28.

We may verify the result by substituting 84 and 28 in the problem.

Thus $\qquad 84 + 28 = 112,$

and $\qquad 84 = 3 \cdot 28.$

PROBLEMS

1. The sum of two numbers is 120. The greater is four times the less. Find each.

2. A certain number plus seven times itself equals 216. Find the number.

3. One number is eight times another. Their sum is 72. Find each.

4. The first of three numbers is twice the third, and the second is four times the third. The sum of the three numbers is 252. Find the numbers.

HINT. Let $x =$ the third number. Then $2x =$ the first, and $4x =$ the second.

5. The sum of three numbers is 117. The second is twice the first, and the third is three times the second. Find each.

6. There are three numbers whose sum is 192. The first is twice the second, and the third equals the sum of the other two. Find the numbers.

7. The sum of three numbers is 312. The second is five times the first, and the third is four times the second. Find the numbers.

8. The sum of three numbers is 208. The second is three times the first, and the third is the sum of the other two. Find the numbers.

9. A man is three times as old as his son. The sum of their ages is 44 years. Find the age of each.

10. The perimeter of a certain square is 160 feet. Find the length of each side.

11. The perimeter of a certain rectangle is 216 feet. It is three times as long as it is wide. Find its dimensions.

12. The perimeter of the rectangle formed by placing two equal squares side by side is 258 inches. Find the side and the perimeter of each square.

5. Literal notation. In algebra numbers are represented by one or more arabic numerals, or by letters, or by both combined.

Thus 3, 25, a, $2b$, $4xy$, and $2x+3$ are algebraic symbols for numbers.

Precisely what numbers $4xy$ and $2x+3$ represent is not known until the numbers for which x and y stand are known. In one problem these letters may have values quite different from those they have in another. To devise methods of determining these values in the various problems which arise is the principal aim of algebra.

6. Factors. A **factor** of a product is any one of the numbers which multiplied together form the product.

Thus $3ab$ means 3 times a times b. Here 3, a, and b are each factors of $3ab$. Similarly, the expression $4(a+b)$ means 4 times the sum of a and b. Here 4 and $a+b$ are factors of $4(a+b)$.

ORAL EXERCISES

1. Name the factors in $3 \cdot 4 \cdot 6$, $2xy$, $3abx$, $4abc$.

In Exercises 2–5, replace a by 3 and b by 4 and find the value of the resulting expression.

2. ab. 3. $3ab$. 4. $2ab$. 5. $5ab$.

7. Exponents. An **exponent** is an integer written at the right of and above another number to show how many times the latter is to be taken as a factor.

(Later this definition will be modified so as to include fractions and other numbers as exponents.)

Thus $3^2 = 3 \cdot 3$; $5^3 = 5 \cdot 5 \cdot 5$. Also $a^4 = a \cdot a \cdot a \cdot a$, and $4xy^3 = 4 \cdot x \cdot y \cdot y \cdot y$. In a^b, b is the exponent of a. If a is 4 and b is 3, $a^b = 4^3 = 4 \cdot 4 \cdot 4$. The exponent 1 is usually not written.

ORAL EXERCISES

1. What are the exponents in $2ac^3$? $3a^2c$? $5a^4x^2$?

2. What is meant by x^2? a^3? b^4? b^5?

3. $4^2=$? 4. $5^2=$? 5. $2^3=$? 6. $3 \cdot 5^2=$?

In Exercises 7–14, replace a by 3 and b by 2 and find the value of the result.

7. a^2. 9. a^3. 11. a^2b. 13. $2a^2$.

8. a^2+b^2. 10. a^3+b^3. 12. a^2b^3. 14. $5a^3b^2$.

8. Coefficients. If a number is the product of two factors, either of these factors is called the **coefficient** of the other in that product.

Thus in $4x^2y$, 4 is the coefficient of x^2y, y is the coefficient of $4x^2$, and $4y$ is the coefficient of x^2. The numerical coefficient 1 is usually omitted. If a numerical coefficient other than 1 occurs, it is usually written first. For instance, we write $5x$, not $x5$.

The following examples illustrate the difference in meaning between a coefficient and an exponent respectively:

$$3x=x+x+x.$$
$$x^3=x \cdot x \cdot x.$$

If $x=5$ in each case, $3x$ stands for the number 15, while x^3 stands for 125. If $x=10$ in each case, $3x=30$, while $x^3=1000$.

ORAL EXERCISES

1. What are the numerical coefficients in $4x$? $5a^2$? $3ax$? $4ac$? $3abc$?

2. What is meant by $3a$? $4x$? $5c$?

3. In $4a^2xy$, name the coefficient of a^2xy, xy, y, a^2x, and a^2y.

9. Use of parentheses and radical signs. If two or more numbers connected by signs of operation are inclosed in parentheses, the entire expression is treated as a symbol for a single number.

Thus $3(6+4)$ means $3 \cdot 10$, or 30; $(17-2) \div (8-3)$ means $15 \div 5$, or 3; $(5+7)^2$ means 12^2, or 144; and $6(x+y)$ means six times the sum of x and y.

As in arithmetic, the symbol for square root is $\sqrt{\ }$, and the symbol for cube root is $\sqrt[3]{\ }$.

The name **radical sign** is applied to all symbols like the following: $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc. The small figure in a radical sign, like the 3 in $\sqrt[3]{\ }$, is often called the **index.**

ORAL EXERCISES

Find the value of:

1. $2(3+4)$.	**6.** $\sqrt{9+7}$.
2. $4(7-2)$.	**7.** $\sqrt{3^2+4^2}$.
3. $(4+3)(5-2)$.	**8.** $\sqrt[3]{4(7-5)}$.
4. $(7-2)(8+3)$.	**9.** $\sqrt[3]{(5+3)(6+2)}$.
5. $\sqrt[3]{8}$.	**10.** $\sqrt{6^2+8^2}$.

NOTE. There has been a considerable variety in the symbols for the roots of numbers. The symbol $\sqrt{\ }$ was introduced in 1544 by the German Stifel, and is a corruption of the initial letter of the Latin word *radix*, which means "root." Before his time square root was denoted by the symbol ℞, used nowadays by physicians on prescriptions as an abbreviation for the word *recipe*. Thus $\sqrt[4]{5}$ would have been denoted by ℞ 4 5. Some early writers used a dot to indicate square root, and expressed $\sqrt{2}$ by $\cdot\, 2$. The Arabs denoted the root of a number by an arabic letter placed directly over the number.

ORAL EXERCISES

1. What are the numerical coefficients in $2x$? $3a^2$? $4xy$? $2ab$? $3\sqrt{a}$?

2. What are the exponents in $3a^2b$? $4a^2b^3$? $5a^3x^4$? $5x^2yz^3$?

3. What is the difference in meaning between the 4 in $4x$ and that in x^4?

4. What is meant by $2x$? $5a$? $8a$?

5. What is meant by $2a^2$? $3x^2$?

6. What is meant by $3(8+6)$? $2(9-4)$? $(7+3)(8-2)$? $(7+3)^2$? $\sqrt{3+6}$? $\sqrt{9+16}$? $\sqrt{100-64}$? $\sqrt[3]{35-8}$? $\sqrt[3]{100-36}$?

7. What is the numerical value of each expression in Exercise 6?

8. Read Exercises 1–16 on pages 15–16.

9. $3 \cdot 5^2 = ?$

10. $(8+2)^3 = ?$

11. $7(6-1) = ?$

12. $(4+3)(5+4) = ?$

13. $(5-1)(8+3) = ?$

14. $3(7-2)(5-3) = ?$

15. $\sqrt{5^2+12^2} = ?$

16. $\sqrt{(10+8)(10-8)} = ?$

In Exercises 17–33, replace a by 3 and b by 4 and find the value of the resulting expressions.

17. $3a^2 - 2b$.
18. $3b^2 + 8a$.
19. $\sqrt{b}$.
20. b^3.
21. $2a^2$.
22. $2a^2b$.
23. $4ab^2$.
24. $2a^2b^2$.
25. $3a^2b^3$.
26. $4ab + b^2$.
27. $\sqrt{3ab}$.
28. $\sqrt{a^2+b^2}$.
29. $\sqrt{3b^2+16}$.
30. $\sqrt{4a^2+7b}$.
31. $\sqrt[3]{2b}$.
32. $\sqrt[3]{ab+15}$.
33. $\sqrt[3]{a^2+b^2-17}$.

EXERCISES

Write, using algebraic symbols:

1. The sum of three times a and four times b.

2. Three times a subtracted from four times b.

3. The square of a subtracted from the square of b.

4. The cube of b subtracted from the square of a.

5. Two times a squared subtracted from three times a squared.

6. The quotient of a and b.

7. The product of four times a squared and b.

8. The sum of a and b divided by their product.

9. The product of a and $2b - c$.

10. The product of a and the sum of b and c.

11. The result of subtracting $a - b$ from $7x$.

12. The sum of the square root of $5a$ and the cube root of $7b$.

13. The product of $x - y$ and the square root of $7x$.

14. The square of the sum of a and b.

15. The square of b subtracted from a.

16. The quotient of three times a multiplied by the square of b, and four times c multiplied by the cube of a.

17. The sum of the quotients of a and $3x$, and $4y$ and c.

10. Order of fundamental arithmetical operations. If we read the expression $6 + 4 \cdot 9 - 12 \div 3$ from left to right, and perform each indicated operation as we come to its symbol, we obtain successively 10, 90, 78, and a final result of 26. If we perform the multiplication and division first, the expression becomes $6 + 36 - 4$, which equals 38. These results show that the value of the expression is determined largely by the order in which the operations are performed. It is customary to observe the

Rule. *In a series of operations involving addition, subtraction, multiplication, and division of arithmetical numbers, the multiplications and divisions shall be performed first, in the order in which they occur. The additions and subtractions in the resulting expression shall then be performed in the order in which they occur or in any other order.*

If parentheses occur, each expression within parentheses should first be simplified in accordance with the preceding rule and the rule then applied to the whole.

EXAMPLES

Simplify:

1. $18 \div 2 + 5 - 4 \cdot 2.$

Solution. $18 \div 2 + 5 - 4 \cdot 2 = 9 + 5 - 8 = 6.$

2. $24 \div 8 \cdot 4 - 6 + 5 \cdot 2 - 7.$

Solution.
$$\begin{aligned} 24 \div 8 \cdot 4 - 6 + 5 \cdot 2 - 7 &= 3 \cdot 4 - 6 + 10 - 7 = \\ &12 - 6 + 10 - 7 = \\ &22 - 13 = 9. \end{aligned}$$

3. $4 \cdot 2 - 3 + 2(8 \cdot 4 - 12 \div 3 + 2 - 6) - 18 \div 2.$

Solution.
$$\begin{aligned} 4 \cdot 2 - 3 + 2(8 \cdot 4 - 12 \div 3 + 2 - 6) - 18 \div 2 &= \\ 8 - 3 + 2(32 - 4 + 2 - 6) - 9 &= \\ 5 + 2(\qquad 24 \qquad) - 9 &= \\ 5 + 48 - 9 &= 44. \end{aligned}$$

EXERCISES

Simplify the following:

1. $20 - 5 + 6 - 10.$
2. $16 - (8 - 2).$
3. $14 - (16 - 8) + (12 - 4).$
4. $6 \div 3 - 2.$
5. $8 \cdot 6 \div 3 - 10.$
6. $18 \div (2 \cdot 3).$
7. $(6 - 3) \cdot (17 - 2 \cdot 5).$
8. $23 - 2 \cdot 6 - 4 \div 2 + 16.$
9. $18 \div (9 - 3).$
10. $(10 - 3) \cdot (16 - 3 \cdot 2 + 8 \div 4).$
11. $14 - 3 \cdot (16 - 2 \cdot 5) \div 6 + 8 \cdot 2.$
12. $(18 - 2) - (4 + 2 \cdot 8 - 18 \div 9) \div 6.$
13. $(16 - 6) \cdot (18 - 8) \div 100 \cdot 5 - 5.$
14. $(5 + 3) \cdot (5 - 3) \div 4 - 3.$
15. $3^2 - 2 \cdot 3 \cdot 1 + 1^2.$
16. $8^2 - 2 \cdot 8 \cdot 3 + 3^2.$
17. $4 \cdot 3 - 2(6 - 2 \cdot 3) + 8.$
18. $8(12 + 4 + 3 - 1).$
19. $3 + 2 \cdot 4 + (3 + 2)4.$
20. $(8 - 2)3 + 8 - 2 \cdot 3.$
21. $3 \cdot 4 - 6 \cdot 0 + 2 \cdot 5 + 2 \cdot 7^2 - 2 \cdot 8.$
22. $(12 + 24 \times 18 \div 3 + 6) \cdot (24 \div 4 + 3 - 2).$

11. Evaluation of algebraic expressions. It is frequently necessary to find the numerical value of an expression for certain values of the letters involved. This process will be found useful in detecting errors made in the solution of equations. (See page 42.)

In practical affairs a working rule, a geometrical relation, or a scientific fact is often stated briefly and conveniently by means of an algebraic expression. Such expressions are frequently called formulas. The student will recall that arithmetic furnished many illustrations of their use. Thus, $A = \frac{ab}{2}$ is the formula for the area of a triangle; $I = P \cdot r \cdot t$ is the formula for simple interest; $A = \pi r^2$ is the formula for the area of a circle, etc.

In finding the numerical value of any literal expression the student should observe the following

Rule. *First, put in place of each letter its numerical value; second, simplify the result thus obtained.*

In any but the simplest expressions the student should *always* observe the two steps of the above rule separately in the order in which they are stated. To mix the two in an attempt to perform mentally both processes at once, is sure to result in many errors and consequent loss of time.

EXERCISES

In Exercises 1–16, let $a = 3$, $b = 1$, $c = 5$, $d = 7$, and $f = 2$. Substitute for each letter its numerical value, and then simplify the results according to the rule of page 13:

1. $4a^2 - 7b$.

Solution.

$$4a^2 - 7b = 4 \cdot 3^2 - 7 \cdot 1$$
$$= 36 - 7 = 29.$$

2. $4a + 3d$.

3. $ab + cd$.

4. $c^2 - 3ab$.

5. $abcd - 5f^2$.

6. $\frac{d+c}{f}$.

7. $\frac{cdf}{3f} + \frac{acf}{2a}$.

8. $\frac{12b}{f} - \frac{c}{b}$.

9. $\frac{1}{c} + \frac{1}{d} + \frac{1}{f}$.

10. $\frac{f^2}{ac} + \frac{6cd}{af}$.

11. $\dfrac{a^2+b^2+c^2+d^2}{a+b+c+d}$.

12. $5f^3+4f^2-4f-5$.

13. $3f^5-9f^4+11f^3-11f^2+13f-20$.

14. af.

15. $df+f^a$.

16. $\dfrac{d+f^c}{3c+a}$.

Find the numerical value of the following expressions when $a=4$, $b=0$, $c=5$, $d=7$, and $f=8$:

17. $\dfrac{4a+3b+2d}{c+f+2}$.

18. $\dfrac{b}{a+c+d}$.

19. $\dfrac{ab}{c}+\dfrac{bd}{a}+\dfrac{bf}{cd}$.

20. $\sqrt{a}+\sqrt{2f}$.

21. $c\sqrt{2af}$.

22. $\sqrt{c^2-a^2}$.

23. $(a+f)\cdot(c+d)$.

24. $ab(a+b)$.

25. $(a+c)\cdot(a+c)$.

26. $a+c(a+c)$.

27. $(a+c)\cdot a+c$.

28. $a+ca+c$.

29. $(f-a)^2$.

30. $(d-a)^3$.

31. $a^2-2ac+c^2$.

32. $a^2+2ab+b^2$.

33. $c^3-3c^2a+3ca^2-a^3$.

34. $d\sqrt[3]{2af}$.

35. $\sqrt{a^2+ac}$.

36. $\sqrt[3]{ad+36}$.

37. If $x=2$ and $y=3$, does $13x-5y=11$?

38. If $x=8$, does $7x-9=3x+25$?

39. Does $x^2-5x+6=0$, if $x=2$? if $x=3$? if $x=4$?

40. Does $x^2-7x+12=0$, if $x=2$? if $x=3$? if $x=4$?

41. Does $2x^2-5x-3=0$, if $x=4$? if $x=3$? if $x=\frac{1}{2}$?

42. The area of a triangle is given by the expression $A=\dfrac{ab}{2}$ in which A is the area, a is the altitude, and b is the base. Find the area of a triangle in which the altitude is 11 inches and the base 14 inches.

43. The area of a circle is given by the formula $A=\pi r^2$ in which A is the area, π is $\frac{22}{7}$ (approximately), and r is the radius. Find the area of a circle whose radius is (a) 7 inches; (b) $\frac{1}{2}$ inch.

44. The volume of a circular cylinder is given by the formula $V=\pi r^2 h$ in which V is the volume, r is the radius of the base, and h is the altitude of the cylinder. Find the volume

of a cylinder in which: (*a*) $r = 3$ inches and $h = 5$ inches; (*b*) $r = 4$ inches and $h = 7$ inches.

45. The distance a body falls from rest is given by the formula $s = \frac{at^2}{2}$ in which $s =$ the distance in feet, $a = 32$, and $t =$ the time in seconds. How far will a body fall in 5 seconds?

46. A balloonist drops a rock while crossing a river. He sees it strike the water 16 seconds later. How high was the balloon at the time?

47. The horse power of a certain kind of gasoline engine is given by the formula $H = \frac{d^2n}{6}$ in which H is the horse power, d is the diameter of each cylinder in inches, and n is the number of cylinders. Find the horse power of a four-cylinder engine in which the diameter of each cylinder is 4 inches.

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS

12. Addition and subtraction. Let us suppose that equal distances are taken on a line and the successive points of division are marked with the natural numbers as follows:

(A)

Such a scale of numbers may be used to illustrate both addition and subtraction as performed in arithmetic.

Thus, in adding 5 to 3 we may begin at 3 and count on 5 spaces to the right, obtaining the sum 8. We shall obtain the same result if we begin at 5 and count on 3 spaces to the right. This process may be stated in general terms thus:

Rule. *To add the number* $\boldsymbol{a}$ *to the number* $\boldsymbol{b}$, *begin at* $\boldsymbol{b}$ *and count on* $\boldsymbol{a}$ *spaces to the right.*

In subtracting 4 from 7 we may begin at 7 and count off 4 spaces to the left, thus obtaining 3. This process may be stated in general terms thus:

Rule. *To subtract the number* $\boldsymbol{a}$ *from the number* $\boldsymbol{b}$, *begin at* $\boldsymbol{b}$ *and count off* $\boldsymbol{a}$ *spaces to the left.*

If we attempt to subtract 5 from 4 by the preceding rule, we arrive at the first point of division to the left of zero. Arithmetic has no number to represent such a result; in fact, the subtraction of 5 from 4 is there regarded as impossible. Arithmetically speaking, such a subtraction cannot be performed. We can, however, subtract 4 of the 5 units from the 4 units, leaving 1 unit unsubtracted.

Now in algebra it is both convenient and necessary to speak of subtracting a greater number from a less, and to call the portion of the greater number which is unsubtracted, the remainder. The fact that such a subtraction is incomplete is indicated by writing a minus sign before the result; thus, $4-5=-1$. Hence the first point of division to the left of zero may be thought of as corresponding to -1. Similarly, $3-5=-2$; and to -2 may correspond the second point of division to the left of zero.

In like manner $5-8=-3$, which corresponds to the third point to the left of zero. In the same way the fourth point of division to the left of zero would correspond to -4, the fifth point to -5, etc.

Such numbers as -1, -2, -3, etc. are called **negative** numbers. The minus sign is never omitted in writing a negative number, though a letter, as x, may denote one.

In opposition to negative numbers the ordinary numbers of arithmetic are called **positive** numbers. If a number has no sign before it, or a plus sign, it is a positive number.

The relative order of positive and negative numbers is indicated in the following scale:

(*B*)

ORAL EXERCISES

Perform the following additions and subtractions by counting along the preceding scale according to the rules on page 18:

1. Add 4 to 3. **3.** Add 6 to -3. **5.** Add 2 to -5.

2. Add 4 to -2. **4.** Add 3 to -3. **6.** Add 5 to -7.

7. Subtract 2 from 5. **10.** Subtract 4 from -3.

8. Subtract 5 from 2. **11.** Subtract 2 from -4.

9. Subtract 6 from 3. **12.** Subtract 2 from -3.

13. Practical use of positive and negative numbers. The scale (B) of positive and negative numbers could be used to measure many of the things with which we come in daily contact. In fact, a practical equivalent is already in use in many instances. Thus, in graduating a thermometer a certain position of the mercury is taken as zero, and the degrees are marked both above and below this point. Hence a temperature reading of 18° is indefinite unless accompanied by the words *above zero* or *below zero*. Usually $+18°$ is taken to indicate the former, while $-18°$ indicates the latter.

Similarly any point on the earth's equator is in zero latitude. Latitude 40° N. means 40° north of the equator. In like manner 30° S. means 30° south of the equator. Obviously $+40°$ and $-30°$ might be used to convey the same ideas.

In general all concrete uses of positive and negative numbers occur in connection with magnitudes which may be regarded as opposite in sense; as, for example, money in bank and an overdrawn account, distances measured in opposite directions from a fixed point, and time measured in the future and in the past from a certain instant.

EXERCISES

1. If the temperature is now $+14°$, what will represent the temperature after a fall (a) of 5°? (b) of 10°? (c) of 18°?

2. If the temperature is now $-16°$, what will it be after a rise (a) of 7°? (b) of 12°? (c) of 25°?

3. In the preceding exercise change the word *rise* to ***fall*** and then answer.

4. A ship sails south from latitude $+13°$ to latitude $-7°$. If one degree is 69 miles, how far did it sail?

5. A ship sails south from latitude $+20°$ at the rate of $5°$ daily. In what latitude is it at the end of each of 6 days? After how many days will it reach latitude $-15°$?

6. A man's property is worth \$4200 and his debts amount to \$2300. How can positive and negative numbers be used to represent (*a*) each of these amounts? (*b*) the man's financial standing?

NOTE. So far as is known the first explanation of positive and negative numbers was by means of the illustration of assets and debts. This is found in the writings of the Hindus before 700 A.D., long before negative numbers were accepted as having any definite meaning. In the use of this illustration the Hindus were nearly a thousand years in advance of the times.

7. If debts and property be reversed in Exercise 6, what would be the answer to (*a*) and (*b*)?

8. The temperature at 6.00 A.M. was $-12°$. During the morning it rose at the rate of $3°$ an hour. What was the temperature at 9.00 A.M.? 10.00 A.M.? 12.00 M.?

14. Addition of positive and negative numbers. As we have seen, subtraction by the use of scale (*B*) is performed by counting spaces to the left. Now a negative number represents an unperformed subtraction; therefore to add a negative number to another number means to perform this subtraction.

For example, in subtracting 8 from 5, -3 was obtained by beginning at 5 and counting 8 spaces to the left, arriving at 3 to the left. Hence when we wish to add -3 to any number, we count off 3 spaces to the left from that number.

To add -8 to 24 we begin at 24 and count off 8 spaces to the left, obtaining 16 as the result; that is,

$$+24+(-8)=+16.$$

Similarly, to add -6 to -4 we begin at -4 and count 6 spaces to the left, obtaining -10 as the result; that is,

$$-4+(-6)=-10.$$

Hence, in general, to *add* a *negative* number n to a given number, begin at the given number and count off n spaces to the *left*.

The **numerical** or **absolute** value of a number is its value without regard to sign.

Thus the absolute values of -3, -5, and $+7$ are 3, 5, and 7 respectively. It should be noted that two different numbers, as $+6$ and -6, may have the same absolute value.

ORAL EXERCISES

By the use of scale (B), page 19, add the following:

1. $+5, +3.$	**4.** $-5, -3.$	**7.** $-6, -2.$	**10.** $-3, -5.$
2. $+5, -3.$	**5.** $7, -3.$	**8.** $6, -4.$	**11.** $-4, +4.$
3. $-5, +3.$	**6.** $-7, -3.$	**9.** $4, -7.$	**12.** $-6, +8.$

The preceding exercises illustrate the correctness of the following working rules:

I. To add two or more positive numbers, find the sum of their absolute values and prefix to this sum the plus sign.

II. To add two or more negative numbers, find the sum of their absolute values and prefix to this sum the minus sign.

III. To add a positive and a negative number, find the difference of their absolute values and prefix to the result the sign of the one which has the greater absolute value.

These rules are of vital importance, as they are of almost constant use throughout the whole of algebra. Rule III is the one in the use of which errors are most likely to occur.

Hence the student will save time if he masters Rule III at *this* point.

The **algebraic sum** of two or more numbers is the number obtained by adding them according to the preceding rules.

The algebraic sum of two numbers is often different from the sum of their absolute values; for example, the algebraic sum of $+9$ and -5 is $+4$, while the sum of their absolute values is 14.

Hereafter the word *add* will mean *find the algebraic sum.*

ORAL EXERCISES

Perform the indicated additions:

1. $+5+(+7)$.
2. $-5+(-7)$.
3. $+5+(-7)$.
4. $-5+(+7)$.
5. $+8+(-5)$.
6. $-8+(+5)$.
7. $-12+(-9)$.
8. $-6+(+6)$.
9. $-6+(-6)$.
10. $-4+(+3)+(+6)$.
11. $3+(-7)+(+5)+(-4)$.
12. $8+(-2)+(-4)+(+6)$.

Answer the questions asked in the following:

13. $7+?=9$.
14. $7+?=2$.
15. $8+?=12$.
16. $8+?=5$.
17. $-8+?=-10$.
18. $-8+?=-6$.
19. $-10+?=-16$.
20. $-10+?=3$.
21. $12+?=5$.
22. $-12+?=4$.

15. Subtraction of positive and negative numbers. If we wish to subtract 7 from 12, we may do so by answering the question, "What number added to 7 gives 12?" By answering a similar question we can subtract 8 from 15, or 25 from 43, or any number a from another number b.

Exercises 13–22, p. 23, are therefore exercises in subtraction, for each asks a question similar to the one in the first sentence of this paragraph.

This point of view brings out the relation that the operation of subtraction bears to that of addition.

ORAL EXERCISES

Perform the following subtractions by answering in each case the question, "What number added to the first number gives the second number?"

Subtract:

1. 5 from 9.
2. 9 from 14.
3. 8 from 5.
4. 13 from 7.
5. -5 from -8.
6. -5 from 9.
7. 5 from -10.
8. -6 from -4.
9. 12 from -18.
10. -25 from 14.

11. In Exercises 1–10, change the sign of the subtrahend (if +, to −; if −, to +) and then add the subtrahend to the minuend. Are the answers the same as were obtained before?

The results obtained in Exercise 11 illustrate the following important principles:

I. Subtracting a positive number is the same in effect as adding a negative number of the same absolute value.

To illustrate: a decrease of $100 in a man's assets is equivalent to an increase of $100 in his liabilities, provided we consider his real financial standing in each case.

II. Subtracting a negative number is the same in effect as adding a positive number of the same absolute value.

To illustrate: a decrease of $75 in a man's liabilities is equivalent to an increase of $75 in his assets, as far as his net financial standing is concerned.

Hence, for the subtraction of positive and negative numbers, we have the

Rule. *Change the sign of the subtrahend (if +, to −; if −, to +). Then find the algebraic sum of the subtrahend (with its sign changed) and the minuend.*

This rule really turns algebraic subtraction into algebraic addition and marks one of the most important distinctions between the operations of arithmetic and those of algebra. For this reason little genuine progress in algebra can be made till the student becomes proficient in the use of the method stated in the rule.

ORAL EXERCISES

Subtract the second number from the first in Exercises 1–16:

1. $8, +5$.	**5.** $14, +9$.	**9.** $+6, +6$.	**13.** $-15, -19$.
2. $+8, -5$.	**6.** $14, -9$.	**10.** $-6, -6$.	**14.** $0, +1$.
3. $+5, -8$.	**7.** $-14, +9$.	**11.** $6, -6$.	**15.** $-1, -1$.
4. $+5, +8$.	**8.** $-14, -9$.	**12.** $-6, +6$.	**16.** $1, -2$.

17. $12-(+3)-(+2)=?$

18. $-10-(+3)-(+2)=?$

19. $-12-(-3)-(-2)=?$

20. $17-(-5)-(+7)=?$

Supply the missing numbers in:

21. $+7+?=10$.

22. $-6+?=-10$.

23. $-5+?=0$.

24. $+6+?=0$.

25. $+4+?=4$.

26. $-8+?=-3$.

27. $-9+?=-5$.

28. $-7+?=7$.

29. $9-?=5$.

30. $-7-?=-5$.

31. $+5-?=-9$.

32. $-7-?=4$.

33. $-5-?=0$.

34. $2-?=0$.

35. $4-?=16$.

36. $6+?=-3$.

Simplify:

37. $12+(7)-(5)$.

38. $12-(-3)+6$.

39. $12-(-4)+(-5)$.

40. $18+(-6)-(+7)$.

41. $-16+(-10)-(+12)$.

42. $-13-(8)+(-14)$.

16. Multiplication of positive and negative numbers. In arithmetic, multiplication was defined as the process of taking one number, the multiplicand, as many times as there are units in another number, the multiplier. The original signification of *times* made this definition meaningless when the multiplier was a fraction; for in $8\times\frac{3}{5}$, 8 could not be added $\frac{3}{5}$ of a time. The definition was then extended and the product of 8 multiplied by $\frac{3}{5}$ was defined to mean 8 multiplied by 3 and the result divided by 5; that is, $8\times\dfrac{3}{5}$ means $\dfrac{8\times 3}{5}$.

Since algebra deals with both positive and negative numbers, we must now extend the arithmetical definition of multiplication and define what sign the product shall have in each of the four cases which may possibly arise:

$$(+4)\cdot(+3)=? \qquad (+4)\cdot(-3)=?$$
$$(-4)\cdot(+3)=? \qquad (-4)\cdot(-3)=?$$

By $(+4)\cdot(+3)$ we mean that $+4$ is to be added three times:

$$(+4)+(+4)+(+4)=+12;$$

that is,

$$(+4)\cdot(+3)=+12.$$

Similarly $(-4)\cdot(+3)$ means

$$(-4)+(-4)+(-4)=-12;$$

that is,

$$(-4)\cdot(+3)=-12.$$

In $(+4)\cdot(-3)$ we mean that 4 is to be subtracted three times. This is the same as subtracting 12 once.

Therefore

$$(+4)\cdot(-3)=-12.$$

Lastly, by $(-4) \cdot (-3)$ we mean that -4 is to be subtracted three times. This is the same as subtracting -12 once, and subtracting -12 once is the same as adding $+12$. Therefore

$$(-4) \cdot (-3) = +12.$$

Summing up,

$$(+4) \cdot (+3) = +12. \qquad (+4) \cdot (-3) = -12.$$
$$(-4) \cdot (+3) = -12. \qquad (-4) \cdot (-3) = +12.$$

Or, in general terms,

$$+a \times +b = +ab.$$
$$-a \times +b = -ab.$$
$$+a \times -b = -ab.$$
$$-a \times -b = +ab.$$

Therefore we have the

Rule. *The product of two numbers having like signs is a positive number, and the product of two numbers having unlike signs is a negative number.*

ORAL EXERCISES

Find the products of the following:

1. $+3, +5$.
2. $+4, +12$.
3. $-5, +6$.
4. $+6, -6$.
5. $-7, +8$.
6. $-7, -4$.
7. $-12, +9$.
8. $+6, -4$.
9. $-7, -6$.
10. $+5, -10$.
11. $+0, +4$.
12. $-7, 0$.
13. $+4, -7, +6$.
14. $+4, -5, -6$.
15. $-4, -5, -3$.
16. $12, +0, -5$.
17. $9, -10, -0$.
18. $-4, +7, -6$.
19. $-3, -2, -5$.
20. $2, -3, +9$.

NOTE. The famous German mathematician Leopold Kronecker (1823–1891) once observed that "the good Lord made the positive integers, but man is responsible for all the rest of the numbers."

This expresses the truth about numbers as accurately as one can in a single sentence. We count objects from our earliest years, and so use the positive integers naturally. It is only when we come to study mathematics that the necessity for any other kind of numbers is forced upon us. Here we see that negative numbers are a great convenience if we wish to represent the relations between objects where oppositeness in any of its many forms is involved. But the artificial character of negative numbers delayed their intelligent use for many hundred years. To be sure, the Hindus said that "the square of negative is positive," but the statement probably did not mean anything to those who read it. It was not until after the time of Descartes (see p. 210) that the rules for operating on negative numbers were understood, even by great mathematicians.

17. Division of positive and negative numbers. When 18 is divided by 9 the result is 2. Here 18 is the dividend, 9 the divisor, and 2 the quotient. The three are connected by the following relation, which holds both for arithmetic and algebra:

$$\text{quotient} \times \text{divisor} = \text{dividend}.$$

We can see that 2 is the correct value of $18 \div 9$, because $2 \times 9 = 18$. This simple test will be applied to determine whether the quotient is a positive or a negative number. All the cases which may arise are represented by the four following questions:

(*a*) $+18 \div +9 = ?$ (*c*) $+18 \div -9 = ?$

(*b*) $-18 \div +9 = ?$ (*d*) $-18 \div -9 = ?$

These questions are answered as follows:

(*a*) $+18 \div +9 = +2$ because $+2 \cdot +9 = +18$.

(*b*) $-18 \div +9 = -2$ because $-2 \cdot +9 = -18$.

(*c*) $+18 \div -9 = -2$ because $-2 \cdot -9 = +18$.

(*d*) $-18 \div -9 = +2$ because $+2 \cdot -9 = -18$.

In (*a*) and (*d*) the dividend and divisor have *like* signs and the sign of the quotient is *plus*. In (*b*) and (*c*) the

dividend and divisor have *unlike* signs and the sign of the quotient is *minus*.

Therefore we have the

Rule. *The quotient of two numbers having like signs is a positive number, and the quotient of two numbers having unlike signs is a negative number.*

The result of multiplication by zero is given a definite meaning in arithmetic and algebra, namely zero; but in both subjects *division by zero is always excluded.* If zero were used as a divisor, numerous contradictions would arise of which the following is an illustration:

Obviously, $0 \cdot 4 = 0,$

and $0 \cdot 6 = 0.$

Therefore $0 \cdot 4 = 0 \cdot 6.$

Dividing each by zero, $4 = 6,$

which is false.

NOTE. The Hindus were the first to express the laws that govern the operations with the number 0. In fact, they were the first to have such a symbol. In the twelfth century a Hindu writer states that $a + 0 = a$, that $\sqrt{0} = 0$, and that $0^2 = 0$. Of course he did not express himself in terms of these symbols, but in the notation of his time and country.

ORAL EXERCISES

Divide the first number by the second in Exercises 1–9:

1. $+10, +2.$

2. $-10, -5.$

3. $-15, +3.$

4. $+14, -2.$

5. $-18, -3.$

6. $-7, +7.$

7. $0, +5.$

8. $0, -5.$

9. $-3, -3.$

10. $+36 \div (+3) \div (-2) = ?$

11. $+45 \div (-5) \div (-3) = ?$

12. $-64 \div (+4) \div (-2) = ?$

13. $+96 \div (-6) \div (+8) = ?$

14. $72 \div (+6) \div (-4) = ?$

15. $60 \div (-5) \div (-12) = ?$

16. $48 \div 2 \div (-4) = ?$

17. $\dfrac{-20}{?} = 4.$

18. $\dfrac{-12}{?} = -2.$

19. $\dfrac{+36}{?} = 4.$

18. Omission of the plus sign before a number. If the first of several numbers connected by either plus or minus signs is a positive number, its sign is omitted; thus $+4-3+6$ is written $4-3+6$. If the sign of the 4 had been negative, the minus sign could not properly have been omitted.

19. Omission of the sign of multiplication. If each of two or more numbers be inclosed in a parenthesis and the parentheses be written one directly after the other with no sign of operation between them, the sign of multiplication is always understood; thus $(6)(3)$ means $6 \cdot 3$. Similarly $(6)(5)(2) = 6 \cdot 5 \cdot 2$.

MISCELLANEOUS ORAL EXERCISES

Simplify the following:

1. $(7)+(5)$.
2. $(7)-(5)$.
3. $(7)+(-5)$.
4. $(7)-(-5)$.
5. $(-7)+(5)$.
6. $-7+5$.
7. $(-9)-(4)$.
8. $-9-4$.
9. $-11+(-13)$.
10. $-6-(-10)$.
11. $-6+10$.
12. $8+(-10)$.
13. $12-18$.
14. $-18-12$.
15. $15-14$.
16. $+7-0$.
17. $0-3$.
18. $(-3)(6)$.
19. $(-5)6$.
20. $(7)(-5)$.
21. $8(-3)$.
22. $(-5)(-12)$.
23. $-3(-8)$.
24. $-5 \cdot 4$.
25. $5 \cdot 0$.
26. $0 \cdot (-9)$.
27. $4 \cdot 8$.
28. $-3 \cdot 6$.
29. $12 \div (-2)$.
30. $-12 \div 2$.
31. $-39 \div (-3)$.
32. $45 \div (-15)$.
33. $0 \div (-6)$.
34. $0 \div 3$.
35. $-27 \div 9$.
36. $3-5+6$.
37. $-4+6-2+1$.
38. $-4+6+2-1$.
39. $2-3+4-5-6$.

Add:

40. $\begin{array}{r} 7 \\ -2 \\ 3 \\ \underline{-5} \end{array}$ **41.** $\begin{array}{r} 6 \\ -2 \\ -3 \\ \underline{4} \end{array}$ **42.** $\begin{array}{r} -8 \\ 6 \\ 2 \\ \underline{-5} \end{array}$ **43.** $\begin{array}{r} 4 \\ -9 \\ -3 \\ \underline{6} \end{array}$

Simplify:

44. $3 \cdot 6 \div 3$.

45. $-4(7) \div (-2)$.

46. $3(-6) \div 2$.

47. $4 \cdot 6(-8) \div (-16)$.

48. $18 \div (-3) \cdot 6 \div 4$.

49. 3^2.

50. $(-3)^2$.

51. 2^3.

52. $(-2)^3$.

53. $(-4)^3 + (4)^2$.

54. $-(-5)^2$.

55. $-(-5)^3$.

56. $-2(3)^2(-2)$.

57. $-3(-2)(-3)^2$

58. $-2(4)^2(-5)$.

59. $2(-4)^2 \cdot 5$.

MISCELLANEOUS EXERCISES

Simplify the following:

1. $4^2 - (-2)^2$.

2. $3^3 - (-2)^3$.

3. $(5-3)(3+2)$.

4. $(6-2)(7-3) \div (4-9)$.

5. $(6-\frac{1}{2})(5+\frac{1}{3})$.

6. $(-1)^2 + (-1)^3 + (-2)^2 + (-2)^3$.

7. $9 + 3 \cdot 2 + 18 \div (-3)$.

8. $5^2 - 4 \div (-2) \div 6(-3)$.

9. $3 \cdot 6 \div 9 - 2 \cdot 6 \div 4 + (-3)^2$.

10. $3^2 + 2 \cdot 3 \cdot (-4) \div (-4)^2$.

11. $(+2)^2 - 2(2)(-3) + (-3)^2$.

12. $3(4)^2(-5) - (-5)^3$.

13. $3^3 + 3(3)^2(-2) + 3(3)(-2)^2 + (-2)^3$.

14. $4^3 - 3(4)^2(-3) + 3(4)(-3)^2 - (-3)^3$.

15. $3^3 - 3(3)^2(0) + 3(3)(0)^2 - 0^3$.

16. $5^3 + 3(5)^2(-4) + 3(5)(-4)^2 - (-4)^3$.

If $x = 3$ and $y = -2$, find the value of:

17. y^2. **19.** y^4. **21.** $2y^2$. **23.** $5x^2y^2$.

18. y^3. **20.** y^5. **22.** $2y^3$. **24.** $4x^2y^4$.

25. $x^2 - y^2$. **28.** $x^2 - 2xy + y^2$.

26. $x^3 - y^3$. **29.** $(x + y)(x - y)$.

27. $x^2 + 2xy + y^2$. **30.** $x^3 + 3x^2y + 3xy^2 + y^3$.

31. Does $4x - 2 = 2x + 8$, if $x = 5$?

32. Does $3x - 5 = 2x + 8$, if $x = -9$?

33. Does $x^2 - x - 12 = 0$, if $x = 4$? if $x = -8$? if $x = -4$?

34. Does $3x^2 + 19x = 14$, if $x = \frac{2}{3}$? if $x = 2$? if $x = -7$?

35. At 7.00 A.M. on a certain day the thermometer registered 27 degrees above zero. The mercury then fell at the rate of 3 degrees per hour. What was the temperature at noon? at 4.00 P.M.? at 5.00 P.M.? When was the temperature -12?

36. A Zeppelin rises 1800 feet from its position, then falls 1200 feet, rises again 1750 feet, falls 400 feet, and then rises 820 feet. How many feet higher or lower is it than at first?

37. Euclid lived about 300 B.C. Sir Isaac Newton died in 1727 A.D. If dates before Christ are considered negative and those after Christ are considered positive, how might these dates be written?

38. What is the meaning of the date -450? of $+1917$? What is the difference in time between the two?

39. A boat is traveling 12 miles per hour. A man on its deck is walking 3 miles per hour. Using positive and negative numbers, represent the rate at which he approaches his destination when he walks toward the bow and when he walks toward the stern.

40. A balloon capable of supporting 500 pounds is held down by 10 men whose average weight is 150 pounds. Using positive and negative numbers, represent the weight of the balloon, of the men, and of the balloon and the men together.

CHAPTER III

ADDITION

20. Monomials. A number symbol which is not the indicated algebraic sum of two or more number symbols is called a **term** or **monomial.**

Thus 5, $-a$, b^4, a^2x, and $-4\,cy^2$ are terms. Frequently, where no confusion would arise, expressions like $(a+b)$, $3(x-y)$, $5\sqrt{x^3}$, and $\sqrt{a-x}$ are called terms, for in such cases one's thought is centered not on the parts of which the expression is composed but on a single number for which the whole stands.

21. Similar terms. Terms that are alike in every respect except their coefficients are called **similar.**

Thus 3, -7, and 9 are similar terms, as well as $\sqrt{2}$ and $3\sqrt{2}$. Also a, $4\,a$, and $-10\,a$ are similar terms, as are a^2x, $-3\,a^2x$, and $7\,a^2x$.

22. Addition of similar terms. We have already learned that $6\,a+3\,a=9\,a$, $6\,a+(-3\,a)=3\,a$, and $5\,xy+6\,xy=11\,xy$. In like manner the sum of $8\,y$, $-3\,y$, $2\,y$, and $-y$ is $6\,y$. The terms $-y$, x, ay, and $-c^2x$ are equivalent to $-1\,y$, $+1\,x$, $+1\,ay$, and $-1\,c^2x$ respectively.

Thus, for adding similar terms we have the

Rule. *Find the algebraic sum of the numerical coefficients and prefix this result to the common literal part.*

ORAL EXERCISES

Find the sum of:

1. $+8, -4$.
2. $+9, -12$.
3. $-6, -9$.
4. $-6\,a, -4\,a$.
5. $+12\,a, -7\,a$.
6. $-9\,a, +3\,a$.
7. $-2\,a, +3\,a$.
8. $+8\,x, -3\,x$.
9. $+5\,x, -7\,x$.
10. $-11\,x, +14\,x$.
11. $-12\,ac, -4\,ac, +9\,ac$.
12. $+ax, -3\,ax, +5\,ax$.

13. $+4\,ax,\ -ax,\ -7\,ax.$

14. $+6\,ax,\ -ax,\ -3\,ax,\ +ax.$

15. $+4\,ac,\ -3\,ac,\ +12\,ac,\ -8\,ac.$

16. $+3\,ac,\ -12\,ac,\ -10\,ac,\ +15\,ac.$

Combine:

17. $ax-2\,ax+8\,ax-3\,ax.$

18. $4\,am-6\,am-10\,am+8\,am.$

19. $5\,am-3\,am+12\,am-7\,am.$

20. $6\,am-7\,am+3\,am-2\,am.$

21. $-3\,ax-4\,ax+7\,ax-12\,ax.$

22. $-ac+4\,ac+ac-2\,ac+12\,ac.$

23. $4\,ax-10\,ax-12\,ax+7\,ax.$

24. $3\,ax-5\,ax+7\,ax-9\,ax+8\,ax.$

23. Order in adding terms. Obviously $2+3=3+2$, and $2-3+5=2+5-3=-3+5+2$, etc. This illustrates the law that in addition the terms may be arranged and added in any order. Hence $6\,d+7\,c=7\,c+6\,d$, and the sum of 3 and x is either $x+3$ or $3+x$; also $a+b=b+a$, and $a+b+c=b+c+a=c+a+b$.

24. Addition of dissimilar terms. Algebraic expressions for the sum of two terms which are not similar, such as $6\,d$ and $7\,c$, are obtained by writing them one after the other with a plus sign between them; thus, $6\,d+7\,c$. The addition of $6\,d$ and $-7\,c$ is indicated by writing $6\,d+(-7\,c)$, or, more briefly, $6\,d-7\,c$. Similarly the sum of $3\,x$, $-2\,y$, and $-7\,z$ may be written $3\,x+(-2\,y)+(-7\,z)$, or, omitting the parentheses and the unnecessary signs, $3\,x-2\,y-7\,z$.

Thus, for adding dissimilar terms we have the

Rule. *Write the terms one after another in any order, giving to each its proper sign.*

If similar and dissimilar terms are to be added, the two preceding rules must both be applied.

EXERCISES

Find the sum of:

1. $a,\ 4\,b,\ -c.$
2. $4\,x,\ -2\,b,\ 5\,y,\ 10.$
3. $3\,ab^2,\ 2\,bx,\ -cy,\ 4\,a^2b.$
4. $5\,x^3y,\ -5\,xy^3,\ c^3y,\ -2\,cy^3.$
5. $6\,x,\ -3\,a,\ 2\,b,\ -5\,x,\ 3\,y.$
6. $5\,a,\ -4\,b,\ +3\,c,\ 7\,b,\ -2\,c^2.$
7. $3\,a^2,\ +2\,b^2,\ -7\,c^2,\ -4\,b^2,\ 5\,a^2.$
8. $5\,a^3b,\ -5\,ab^3,\ -3\,a^2b,\ 3\,ab^2,\ 4\,a^3b,\ 2\,ab^3.$

Simplify:

9. $11+5-9-3+16-25.$
10. $16\,ac-9\,xy+5\,ac-2\,xy-6\,ac+11\,xy.$
11. $7\,x+6\,a-15\,a-8\,x+3\,x.$
12. $12\,y-17\,y+10\,b+20\,y-25\,b.$
13. $4\,ab-8\,xy-12\,ab+15\,ab-xy.$
14. $14\,x^2-13\,y^2+x^2-5\,x^2+y^2.$
15. $5\,z^2-4\,y^2-11\,y^2+z^2-7\,y^2.$
16. $7\,c^2d-5\,a^2b+6\,c^2d-a^2b+9\,a^2b.$
17. $-4\,a^2b^2+6\,x^2y^2-15\,a^2b^2+3\,x^2y^2+0\,a^2b^2.$
18. $-b^3-19\,a+17\,b^3+b^3-0\,b^3+13\,a.$
19. $12\,a^3b+6\,c^2d-a^3b+16\,a^3b-13\,a^3b-25\,c^2d.$
20. $11\sqrt{a}-16\sqrt{a}+21\sqrt{a}-\sqrt{a}.$
21. $3\sqrt{x-y}-\sqrt{x-y}+8\sqrt{x-y}-7\sqrt{x-y}.$
22. $3(a+b)-5(a+b)+8(a+b).$
23. $-7(a-2\,b)+(a-2\,b)+12(a-2\,b).$

25. Addition of polynomials. A **polynomial** is an algebraic expression consisting of two or more terms.

It is not usual to call an expression a polynomial if any of its terms contain a letter under a radical sign. Thus we shall not call expressions like $\sqrt{x-3}+4$ polynomials.

A **binomial** is a polynomial of two terms.

A **trinomial** is a polynomial of three terms.

EXAMPLE

Add the following polynomials: $4a - 6b - a^2c$; $3b + 4a^2c$; $-3a - 7a^2c + 10$; $5a + 3b - 6$.

Solution.

$$\begin{array}{rrrrr} & +4a & -6b & -a^2c & \\ & & +3b & +4a^2c & \\ & -3a & & -7a^2c & +10 \\ & +5a & +3b & & -6 \\ \hline \textit{Sum,} & 6a & & -4a^2c & +4 \end{array}$$

For the addition of polynomials we have the

Rule. *Write similar terms in the same column.*

Find the algebraic sum of the terms in each column and write the results in succession with their proper signs.

A **check** on an operation is another operation which tests the correctness of the first.

For example, in arithmetic the result of division is checked by multiplication; thus the check for $132 \div 6 = 22$, is $22 \cdot 6 = 132$.

Addition in arithmetic is usually checked by adding the columns in the opposite direction. In the two cases the mind deals at every step with different numbers, and if the final result is the same, its correctness is more probable than after either addition alone. But for arithmetical addition, and for many of the operations of algebra, no really satisfactory check exists.

In elementary algebra long columns in addition do not occur. Where columns contain three or more numbers they may be added term by term upward or downward. If the numbers in a column are not all of the same sign, we can partially check results by adding the positive and the negative numbers in such columns separately and finding the algebraic sum of the two results.

EXERCISES

Add the following polynomials and check the results:

1. $x + 3$, $2x - 6$, and $3x - 5$.
2. $x - y - 3$, $2x + y - 8$, and $7x - 4y + 10$.
3. $x + y + z$, $x - 3y + 4z$, and $3x + 4y - 7z$.
4. $2x + 5y - z$, $3x - 8y + 6z$, and $x - y - z$.
5. $3x - 1$, $4 - 2x$, $6x - 8$, and $3x + 3$.
6. $x^2 - 9$, $16 - 4x^2$, $8x^2 + 4$, and $3 - 2x^2$.
7. $5x + 5y$, $4x - 9y + 6z$, and $3x - 3y - 3z$.
8. $7x - 6y + 3z$, $5x - 4z$, and $2x + 6y - 5z$.
9. $x^2 - x - 2$, $3x^2 - 4x + 5$, and $5x^2 + 8x - 7$.
10. $x^2 - 4x + 1$, $3x - 2x^2 + 8$, and $10x - 11x^2 - 18$.
11. $3c - 4c^2 - 5$, $c^2 + c + 1$, $4c - 2c^2 + 3$, and $5 - 2c - 5c^2$.
12. $4x - 5y - 5z$, $6y - 2z$, and $7x - 6y - 9z$.
13. $x + 2z + 3y$, $y - 3z + x$, and $z - 2x - 4y$.
14. $5x - 6y + 7z$, $2y - 11z + x$, and $9z - 5y$.
15. $8a - 7b - 6c$, $5c - 4a - 3b$, and $3b + 7c$.
16. $9ac - bc$, $7ab - 3ac$, and $-12ab - ac$.
17. $a^2 - 4a + 10$, $5a - 6a^2 + 4$, and $3a - 16 + 2a^2$.
18. $9 - a^2 + 3a$, $-13 + 6a^2 - 4a$, and $3a^2 - a$.
19. $a - x - 3c$, $4 - 7x + 3a$, $7c - 2a + 10$, and $a - c - 2x - 11$.
20. $a + x + c + 5$, $2a + 2c + 10 + 2x$, $a - c - 7 - x$, and $7 - 3x - 2a + 4c$.
21. $x - 1 - 3c$, $x - a - 2c$, $a - 3c - 5$, and $2x - 7 + 11a$.
22. $2x + 3y + 5z$, $x - y + 3z$, and $3x - 2y + 10z$.
23. $4x - 20z$, $8x + 9y$, and $5z - 3y$.
24. $a - 3(x - y) + z$, $5 - 10a + 4(x - y)$, and $-2(x - y) + 6$.
25. $a + d + 2(b - c)$, $7(b - c) + 6d - 12a$, and $11a - 5(b - c)$.

Combine similar terms in the following polynomials:

26. $7a^2 - 13b^2 + 12c^2 + 15b^2 - a^2 - 7c^2 + 3b^2 + 5c^2.$

27. $x^2 + 2xy + y^2 - 5xy - 4x^2 - y^2 + 16x^2 - 8xy + y^2.$

28. $5ab - a^2 + b^2 + 4ab - 9b^2 + 5a^2 - 2ab - 2b^2.$

29. $5x^2 - 6x + 11 - 4x - 8 - 3x^2 + 7x - 18 + 13x.$

30. $12c^2 - 10bc + 8b^2 + bc - 6b^2 - c^2 + c^3 - 11c^3.$

31. $4x^2y - xy + y^2 - 3xy + 4xy^2 - 2x^2y + 4y^2 + 3xy^2 + x^2y.$

32. $\frac{1}{2}a + \frac{1}{3}b - \frac{1}{4}c + c + \frac{1}{2}b - \frac{2}{3}a - b + \frac{1}{2}c + \frac{3}{4}a + 7.$

The sum of $5x$ and $2x$ may be written $(5+2)x$. This is not usual or necessary, as 5 and 2 can be combined and the result written $7x$. In adding $5x$ and ax, however, the 5 and the a cannot be combined and the result expressed by a single character, so the sum is written $(5+a)x$. Similarly, $ax - 3x = (a-3)x$, $ax + x = (a+1)x$, and $ax - bx + x = (a - b + 1)x$. Also $a(x+y) + b(x+y) = (a+b)(x+y)$.

Write so that x will have a polynomial coefficient:

33. $ax + 3x.$ **35.** $3cx - x.$ **37.** $2ax - 3x + bx.$

34. $2ax + x.$ **36.** $ax + bx + cx.$ **38.** $3ax - 4cx + x.$

39. $3ax - bx - x + a^2x.$ **40.** $bx - 5cx - x - 4bx.$

Write so that the binomial will have a polynomial coefficient:

41. $a(b+c) + 3(b+c).$ **42.** $4(a-x) - 5b(a-x).$

43. $8a(a+3b) - 1(a+3b).$

44. $7b(x^2+y^2) - a(x^2+y^2) + (x^2+y^2).$

CHAPTER IV

SIMPLE EQUATIONS

26. Definitions. An **equation** is a statement of equality between two equal numbers or number symbols.

Thus $2 = 5 - 3$, $a - 2b = 3a + b - 2a - 3b$, $4x = x + 12$, and $x^2 - 5x + 6 = 0$ are equations.

The part of an equation on the left of the equality sign is called the *first* or *left* member, that on the right, the *second* or *right* member.

In an equation a letter whose value is sought is called the *unknown letter*, or simply the *unknown*.

The process of finding the value of the unknown letter in an equation is called *solving the equation*.

27. Axioms. An **axiom** is a statement whose truth is accepted without proof.

In the solution of equations constant use is made of four axioms.

Axiom I. *If the same number is added to each member of an equation, the result is an equation.*

Thus, adding 5 to each member of $x - 5 = 7$ gives $x = 12$. Axiom I states that if this addition is performed, the result is true. Hence the original equation is solved.

Axiom II. *If the same number is subtracted from each member of an equation, the result is an equation.*

Thus, subtracting 4 from each member of $x + 4 = 10$ gives $x = 6$. Axiom II states that if this subtraction is performed, the result is true.

Axiom III. *If each member of an equation is multiplied by the same number, the result is an equation.*

If an equation is in the form $\frac{x}{2} = 6$, it can be solved by multiplying each member by 2, giving $x = 12$. Axiom III states that if this multiplication is performed, the result is true.

Axiom IV. *If each member of an equation is divided by the same number (not zero), the result is an equation.*

If an equation is in a form like $3x = 12$, it can be solved by dividing each member by the coefficient of x. Thus, dividing each member of $3x = 12$ by 3, we get $x = 4$. Axiom IV states that if this division is performed, the result is an equality.

If all terms containing the unknown letter are in one member and all numerical terms in the other, the like terms may be united and the equation solved.

Thus $5x - 2x + x = 8 + 15 - 3$ becomes, when like terms are united, $4x = 20$, and dividing each member by 4, we obtain $x = 5$.

Usually, numerical terms, as well as terms containing the unknown letter, will be found in each member of an equation, as in $5x + 3 = 2x + 18$. By the use of one or more of the preceding axioms it is always possible to change the form of such equations until they are similar to the equation $3x = 15$, which, as we have seen, can easily be solved.

In changing the form of an equation, by an application of the foregoing axioms, it is important to note that we do not change the value of the unknown. We merely discover the value which it really had all the time. In fact, the chief significance of the foregoing axioms lies in their application to any kind of equation, whether it involves numbers, letters standing for known numbers or for unknown numbers, or all of these together.

ORAL EXERCISES

Find the value of the unknown in each of the following equations. State the axiom or axioms used in solving each:

1. $2x = 4$.
2. $3x = 12$.
3. $5y = 15$.
4. $2n = 16$.
5. $7x = 35$.
6. $x + 1 = 2$.
7. $k + 3 = 5$.
8. $p + 5 = 11$.
9. $x + 2 = 17$.
10. $y + 9 = 16$.
11. $y - 1 = 1$.
12. $x - 2 = 3$.
13. $z - 4 = 5$.
14. $x - 5 = 1$.
15. $h - 7 = 3$.
16. $\frac{x}{2} = 1$.
17. $\frac{n}{2} = 3$.
18. $\frac{x}{3} = 3$.
19. $\frac{y}{5} = 4$.
20. $\frac{x}{7} = 2$.
21. $x + 4 = 7$.
22. $n - 2 = 6$.
23. $5x = 30$.
24. $y + 3 = 11$.
25. $y - 6 = 3$.
26. $\frac{x}{6} = 4$.
27. $\frac{n}{4} = 4$.
28. $7x = 7$.
29. $\frac{x}{7} = 7$.
30. $2x + 1 = 3$.
31. $2q + 5 = 11$.
32. $5n - 9 = 6$.
33. $3x = x + 4$.
34. $2x + 1 = 2$.

EXAMPLE

Solve $5x - 8 = 2x + 19$.

Solution.	$5x - 8 = 2x + 19$.	
Subtracting $2x$ from each member,	$3x - 8 = 19$.	Ax. II
Adding 8 to each member,	$3x = 27$.	Ax. I
Dividing each member by 3,	$x = 9$.	Ax. IV

This method of solution illustrates the following

Rule. *By the applications of Axioms I and II change the equation so that all of the terms containing the unknown number are in one (usually the left) member of the equation, and all other terms in the other member.*

Combine the terms in each member.

Divide each member by the coefficient of the unknown.

Checking the solution of an equation is often called *testing* or *verifying* the result. For this we have the

Rule. *Substitute the value of the unknown obtained from the solution in place of the letter which represents the unknown in the original equation. Then simplify the result until the two members are seen to be identical.*

Check. $5x - 8 = 2x + 19.$

Substituting 9 for x, $5 \cdot 9 - 8 = 2 \cdot 9 + 19.$

Simplifying, $45 - 8 = 18 + 19,$

or $37 = 37.$

EXERCISES

Find the value of the unknown in each of the following equations and verify results:

1. $x + 5 = 7.$

2. $y - 9 = 6.$

3. $4x - 6 = 14.$

4. $4x + 2 = 3x - 4.$

5. $5x - 2 = 4x + 5.$

6. $5p = 3p + 8.$

7. $4x + 2 = 2x.$

8. $3n = n + 3.$

9. $4x + 2 = x + 5.$

10. $5x + 6 = 3x + 10.$

11. $4x - 7 = 2x + 3.$

12. $3k - 5 = 7 - k.$

13. $6 - a = 9 - 4a.$

14. $2x = 1 - x.$

15. $5 + 3y = y + 4.$

16. $2x + 8 = 5x - 1.$

HINT. Subtract $2x$ from each member, etc.

17. $5 + 2q = 5q - 1.$

18. $2 - 3n = n + 1.$

19. $4x - 2 = 7 + x.$

20. $4x + 3 - x + 5 = x - 10.$

21. $3 + 2z - 1 = 12 - 3z.$

22. $2x + 3x + x = x + 9 + 2x.$

23. $4y - 3 + 3y - 4 = 6y - 8.$

24. $36 - 8x + 4 - 9x = 2x + 10.$

25. $3x + 12x + 17 - x + 4x = 107.$

26. $4k - 3 = 5k - 16 - 43k - 71.$

ORAL EXERCISES

1. A rope is 10 feet long. How long will it be if 3 feet are cut off one end?

2. If 3 feet are cut off one end of a rope x feet long, how much remains?

3. A river is d feet deep at a ford, and 8 feet deeper under a bridge. How deep is it under the bridge?

4. A 25-gallon tank is full of gasoline. If g gallons are pumped out, how many gallons are left?

5. A 50-gallon tank is full of gasoline. If gasoline is pumped out until there are only x gallons left in the tank, how many gallons are removed?

6. If the sum of two numbers is 100, and one of them is n, what is the other?

7. A pole y feet long is cut into two pieces. If one piece is f feet long, how long is the other?

8. If the sum of two numbers is s and one of them is 10, what is the other?

9. A certain chimney is three times as high as a neighboring telegraph pole. If the pole is h feet high, how high is the chimney?

10. A rectangle is three times as long as it is wide. If it is x feet wide, how long is it? What is its perimeter?

11. There are twice as many boys in a certain class as there are girls. If there are n girls, how many boys are there? How many pupils all together? How many people in the room, including the teacher?

12. The roof of a certain building is five times as far from the ground as the bottom of the third-story windows. If the windows are y feet from the ground, how high is the roof? How long would a wire have to be to reach from the roof to the ground and back to a third-story window? How long would it have to be to reach from the roof to the bottom of the window direct?

13. John's father is twice as tall as John, and 6 inches taller than John's mother. If John is x inches tall, how tall is his father? his mother?

Even integers are those exactly divisible by 2. Odd integers are those not exactly divisible by 2.

Consecutive integers are integers arranged in the natural order, like 6, 7, 8, 9, 10, etc.

Consecutive odd integers are odd integers arranged in the natural order, like 5, 7, 9, 11, 13, etc.

Consecutive even integers are even integers arranged in the natural order, like 6, 8, 10, 12, 14, etc.

14. What is the difference between any two consecutive integers? between any two consecutive odd integers? between any two consecutive even integers?

15. If n is an integer, what is the next consecutive integer? If x is an integer, what are the next two consecutive integers?

16. If x is an odd integer, what is the next consecutive odd integer? the next two consecutive even integers?

17. If e is an even integer, what is the next consecutive even integer? the next three consecutive odd integers?

18. If x is an odd integer, is $x+1$ odd or even? If x is even, what of $x+1$?

19. If x is an odd integer, what of $x+3$? $x+6$? If y is even, what of $y+2$? $y+3$?

20. If x is an integer, is $2x$ odd or even? Is $2x+1$ odd or even?

21. Is the sum of two consecutive integers always odd?

Hint. Express the integers algebraically, add them, and divide by 2.

22. Is the sum of three consecutive integers always even?

23. By what number is the sum of three consecutive integers always divisible?

Express the following statements as equations:

24. The sum of 5 and x is 8.

25. The sum of x and 4 is 9.

26. x is equal to 5 diminished by 1.

27. x is equal to 10 diminished by 3.

28. x is 4 more than 6.

29. Three times x is 15.

30. Four times a diminished by 2 equals 18.

31. x is 6 less than 9.

32. Three times y is greater by 4 than 10.

33. One half of x increased by 5 equals 9.

34. Two thirds of x increased by 6 equals 14.

35. 8 added to x gives the same result as x taken from 22.

36. $2x$ diminished by 9 is equal to x increased by 14.

37. If 6 is taken from three times x, the result is the same as when x is taken from 14.

28. Solution of problems. In the solution of problems in simple equations the following steps are necessary:

1. *Read the problem carefully and find the statement which will later be expressed by the equation.*

2. *Represent the unknown number by a letter.*

3. *Express the conditions stated in the problem as an equation involving this letter.*

4. *Solve the equation.*

5. *Check by substituting in the problem the value found for the unknown.*

In the preceding sentence the words "in the problem" are of importance, for substituting the value found in the equation would not detect any errors made in translating the words of the problem into the equation.

The foregoing directions for the solution of the various problems leading to simple equations are as definite as can be given. The student will obtain much aid from the study of the typical solutions which occur from time to time. Then one or more careful readings of each problem, a little fixing of the attention upon it, and an application of common sense will insure progress.

EXAMPLE

A man wishes to give 75 cents to his two children so that the older receives 15 cents more than the younger. How much shall he give to each?

Solution. The amount the older child receives + the amount the younger child receives = 75 cents. Let x denote the number of cents the younger child receives. Then $x + 15$ denotes the number of cents the older child receives.

Substituting these symbols in the statement above, we have

$$x + 15 + x = 75.$$

$$2x + 15 = 75.$$

$$2x = 60.$$

$$x = 30.$$

$$x + 15 = 45.$$

Hence the younger child receives 30 cents, and the older receives 45 cents.

Check. $75 = 45 + 30$, and $45 - 30 = 15$.

PROBLEMS

1. One boy sells 5 more newspapers than another. Together they sell 67 papers. How many does each boy sell.

2. It is desired to divide a class of 51 students into two groups, one of which shall contain 3 more than the other. How many will each group contain?

3. One number is five times another, and their sum is 30. Find both numbers.

4. The sum of two numbers is 72, and the greater is three times the less. Find both numbers.

5. One number is five times another, and their difference is 52. Find both numbers.

6. A freight train contains 72 cars. It is desired to divide it into two trains, of which one shall contain twice as many cars as the other. How many cars will there be in each train?

7. The perimeter of a certain square is 24 feet. Find the length of each side.

8. A rectangle is three times as long as it is wide, and its perimeter is 64 feet. Find its length and its width.

9. The perimeter of a certain rectangle is 78 feet. Its length is five times its breadth. Find its dimensions.

10. A rectangle is 12 feet longer than it is wide, and its perimeter is 96 feet. Find its length and its width.

11. It is desired to cut a log into two pieces so that one piece shall be 8 feet longer than the other. If the log is 36 feet long, how long will each piece be?

12. A stick 120 inches long is to be cut into two pieces, one of which is to be 14 inches more than three times the length of the other. How long is each piece?

13. The sum of three numbers is 22. The second is twice the first, and the third is four times the second. Find each number.

Hint. In solving a problem involving three or more unknown numbers the student should read the statement carefully and decide which of the numbers to be found he should indicate by x. He should select for this purpose the number in terms of which the other numbers are most easily expressed.

14. The first of three numbers is twice the third, and the second is four times the third. The sum of the three numbers is 63. Find each number.

15. It is desired to divide a company of 95 soldiers into three squads of which the second contains 5 more men than the first, and the third contains 10 more than the second. How many men will there be in each squad?

16. It is desired to divide a train of 62 cars into two sections, one of which is to contain 2 more than twice as many cars as the other. How many cars will there be in each section?

17. Find two consecutive numbers whose sum is 45.

18. Find two consecutive odd numbers whose sum is 36.

19. Find three consecutive numbers whose sum is 72.

20. Find three consecutive even numbers whose sum is 48.

21. Find five consecutive odd numbers whose sum is 85.

22. It is desired to cut a stick 36 inches long into three pieces whose lengths are consecutive integers. How long will the pieces be?

Biographical Note. *John Wallis.* Among those who introduced and helped to standardize the modern algebraic symbolism was John Wallis, an Englishman. He was the son of a clergyman, and, like most scholars of his day, did not confine his interest to any one subject. He was at various times an instructor in Latin, Greek, and Hebrew, and for many years was professor of mathematics at Oxford. He also invented a method of teaching deaf mutes to talk.

During the wars between Charles I and Cromwell, Wallis's sympathies were with Cromwell, and he was of great service in reading royalist dispatches written in cipher. In fact, he was one of the most famous cryptologists of his day.

Wallis did not become interested in mathematics till the age of thirty-one, but devoted himself to the subject for the rest of his life. One of the earliest and most important books on algebra ever written in English was his treatise published in 1685. It contains a brief historical sketch of the subject which is unfortunately not entirely accurate, but his treatment of the theory and practice of arithmetic and algebra has made the book a standard work for reference ever since.

JOHN WALLIS

CHAPTER V

SUBTRACTION

29. Subtraction of monomials. The principles stated on page 25 apply to the subtraction of monomials as well as of the positive and negative numbers there used. Hence for subtracting one monomial from another we have the

Rule. *Change the sign of the subtrahend; then find the algebraic sum of this result and the minuend.*

As soon as possible the student should learn to change the sign of the subtrahend *mentally.*

EXAMPLES

1. From $+8a$ take $+5a$.

Solution. $+8a$ minus $+5a = 8a - 5a = 3a$.

2. From $6ax$ take $-2ax$.

Solution. $6ax$ minus $-2ax = 6ax + 2ax = 8ax$.

3. Subtract $7a^2b$ from $-10a^2b$.

Solution. $-10a^2b$ minus $7a^2b = -10a^2b - 7a^2b = -17a^2b$.

The difference of two dissimilar monomials cannot be written as a single term, but is expressed by a binomial, as follows:

4. Subtract $+a$ from $+b$.

Solution. $+b$ minus $+a = b - a$.

5. Subtract $-4b$ from $3c$.

Solution. $3c$ minus $-4b = 3c + 4b$.

6. Subtract $4xy$ from $-5x^2z$.

Solution. $-5x^2z$ minus $4xy = -5x^2z - 4xy$.

ORAL EXERCISES

In each of the following, subtract the second term from the first:

1. $8a, 3a.$
2. $9a, 6a.$
3. $11a, 14a.$
4. $7a, 12a.$
5. $4a, -2a.$
6. $6a, -5a.$
7. $5c, -7c.$
8. $4c, -9c.$
9. $-3a, 6a.$
10. $-5a, 8a.$
11. $-7a, -10a.$
12. $-11x, 6x.$
13. $-6a, -4a.$
14. $-8a, -6a.$
15. $-4a, -7a.$
16. $-3a, -8a.$
17. $-9x, -12x.$
18. $12x, -5x.$
19. $-4ac, -4ac.$
20. $-4ac, 4ac.$
21. $4ac, -4ac.$

Subtract the first monomial from the second, and also the second monomial from the first, in each of the following:

22. $2x, 5x.$
23. $4x, 7x.$
24. $-2x, -3x.$
25. $-5x, -3x.$
26. $-x, 4x.$
27. $-x, -3x.$
28. $-3c, 5c.$
29. $-ac, -5ac.$
30. $8a^2c, -11a^2c.$
31. $5x^2y^2, -5x^2y^2.$
32. $-4ab, 0.$
33. $x, y.$
34. $c, 3x.$
35. $x, -5y.$
36. $-4a, 6b.$
37. $-2a, -5b.$
38. $-2a, -2a.$
39. $5x, -5x.$

30. Subtraction of polynomials. For the subtraction of polynomials we have the

Rule. *Write the subtrahend under the minuend so that similar terms are in the same column.*

Then, changing the sign of each term of the subtrahend mentally, apply the rule on page 49 to each column.

Check. *In algebra, as in arithmetic, work in subtraction is checked by use of the relation*

$$\textbf{difference} + \textbf{subtrahend} = \textbf{minuend}$$

EXAMPLE

Subtract $5x - 2y - 7z + 2$ from $3x + 8y - 5z$, and check the result.

Solution.

$$\begin{array}{rrrrl} 3x + & 8y - & 5z & & = \text{minuend} \\ 5x - & 2y - & 7z & + 2 & = \text{subtrahend} \\ \hline -2x + & 10y + & 2z & - 2 & = \text{difference} \end{array}$$

Check.

Difference + subtrahend $\quad 3x + 8y - 5z = \text{minuend}$

EXERCISES

In Exercises 1–12, subtract the first number from the second:

1. $a + 2,\ a + 3.$
2. $a - 4,\ a - 6.$
3. $2a - 3,\ 3a + 7.$
4. $3a + 7,\ 9 - 4a.$
5. $2x - 5,\ 5 - 2x.$
6. $6x - 3,\ 6 + 3x.$
7. $9x + 4,\ 4x - 9.$
8. $4a,\ 4a - 2.$
9. $2ab - 5c,\ 5c.$
10. $3xy - a,\ 5xy.$
11. $0,\ x + 4.$
12. $2x - 5,\ 0.$

In Exercises 13–26, subtract the first polynomial from the second and check the work:

13. $x - 2y - 3z,\ 2x - 2y + z.$
14. $4x - 8y + 2z,\ 4x - 5y - 3z.$
15. $3a - 2b,\ 4a + b + 2c.$
16. $5a - 4b + 3c,\ 3a - 5b.$
17. $3x^2 - x - 5,\ 5x^2 + 8x - 2.$
18. $7x^2 - 4x + 11,\ 8x - 5x^2 - 13.$
19. $a - x + y,\ b - x - y.$
20. $3a - 2b + c + 6,\ 4a - b + 5c.$
21. $2a + 2b - 2c - 4,\ a - 3b.$
22. $2a - 3x + 4c,\ 4x - 3a - 3c - 11.$
23. $a + b - c,\ c - a - b.$
24. $2a^2 - 3ab + b^2,\ 3b^2 - 7a^2 + 8ab.$

25. $3x^3 - 3x^2y - 3xy^2$, $4x^2y - 5xy^2 - y^3$.

26. $a^3 - c^3 - 3a^2c + 3ac^2$, $5a^2c + 4c^3 + 3a^3 + 2ac^2$.

In Exercises 27–35, find the expression which added to the first will give the second:

27. $x - 2y + z$, $2x + 5y - 3z$.

28. $7x - 9y - 3z$, $5x - 2y - 4z$.

29. $5a - 4b + 6c$, $6a - 3b$.

30. $a - b + 2c$, 5.

31. $ax^2 + bx + c$, $2bx - c$.

32. $3ab + c$, $3xy - z$.

33. $2x - 4y - z$, 0.

34. $a^2 - 2ac + 3c^2$, $2c^2 - 3ac + 4a^2$.

35. $2x^2y - 3xy^2$, $y^2x + yx^2 - z$.

In Exercises 36–42, find the expression which subtracted from the first will give the second:

36. $x - 2y - z$, $3x - 2y + z$.

37. $x^2 - 7x - 10$, $14x - 8 + 3x^2$.

38. $x - y - z$, $5x + 3y - 8z$.

39. $4 - 8x^2$, $x^2 + 5x - 6$.

40. $a^4 + a^2c^2 + c^4$, $2c^4 - 3a^2c^2 - 4a^4$.

41. $3a + 5b - c$, 0.

42. $4a + 6b - 8y$, $a - 12$.

43. Subtract the sum of $a^2 + 2ab - 1$ and $a^2 + 12ab - 20$ from $a^2 + 13ab - 30$.

44. Subtract the sum of $a - 3b + c$ and $4a + 5b - 6c + 4$ from $a - b + c - x$.

45. From the sum of $5x + 3x^2y - 15xy^2$ and $-6x - 12xy^2 + 7yx^2$ subtract $11x - 5x^2y + 7y^2x$.

46. From the sum of $4abc^2 - 3ab^2c + 2a^2bc$ and $6abc^2 - 5ab^2c - 4a^2b^2c$ subtract $2abc^2 - 3a^2bc + 7ab^2c$.

47. From the sum of $3x - 4xy - 2z$ and $7xy - 4z - 3x$ take the sum of $5z - 2xy - a^2bc$ and $9x - 6a^2bc - z$.

Find the algebraic sum of:

48. $(4x - 3y + 6) + (3x + 5y - 10)$.

49. $(7c + 5d - e) - (4c + 5d - 9e)$.

50. $(x^2 + 2x + 5) + (2x^2 + x - 10) - (x^2 - 5x + 3)$.

51. $(x + 3y - 2z) + (4x - 5y + 3z) - (3x - 2y - 6z)$.

52. $(5x + 3y - z) + (4y + 7z) - (x - y + 3z)$.

53. $4x - 3y + 7 - (2x - 5y - 4) + (4x - 8)$.

54. $3c - 5d - e - (5c + 6d + 11e) - (5c + 4e)$.

55. $3a + 3b - 4c - (-3b - 3c - 4) - (4a + x - 8c)$.

56. $x^3 - c^3 + 3cx^2 - 3c^2x - (x^3 + c^3 - 2c^2x - 4cx^2)$.

CHAPTER VI

IDENTITIES AND EQUATIONS OF CONDITION

31. Kinds of equations. Equations are of two kinds: **identities** and **equations of condition.**

An **identity** is an equation in which, if the indicated operations are performed, the two members become precisely alike, term for term.

Thus, $4 \cdot 5 + 3 \cdot 4 = 8 \cdot 5 - \frac{4 \cdot 6}{3}$ is an identity, for, if we perform the indicated operations, it becomes $20 + 12 = 40 - 8$, or $32 = 32$.

Similarly, $2a + 3b - 4 = 3a - 2b + (5b - a - 4)$ is an identity, for, if we perform the indicated addition in the second member, the equation becomes $2a + 3b - 4 = 2a + 3b - 4$, in which the two members are alike, term for term.

An identity which involves letters is true for *any* numerical values of the letters in it.

Thus the literal identity $(a + 3)^2 = a^2 + 6a + 9$ becomes, when $a = 5$, $(5 + 3)^2 = 5^2 + 6 \cdot 5 + 9$, or $8^2 = 25 + 30 + 9$, or $64 = 64$. If a is zero, the identity becomes $(0 + 3)^2 = 0 + 6 \cdot 0 + 9$, or $9 = 9$.

An equation in one unknown which is true only for certain values of the unknown is an **equation of condition,** which for brevity we shall usually refer to simply as an equation.

The statement $4x = x + 12$ is true only when $x = 4$. If 4 is substituted for x, the equation becomes the identity $4 \cdot 4 = 4 + 12$, or $16 = 16$. Clearly the statement is false if 0, or 3, or any value other than 4 is put for x; it is true on condition that x be 4, and on no other.

Similarly, $x^2 - 5x + 6 = 0$ is true when $x = 2$ or when $x = 3$. In the first case $x^2 - 5x + 6 = 0$ becomes $2^2 - 5 \cdot 2 + 6 = 0$, or $4 - 10 + 6 = 0$, or $0 = 0$. In the second case we obtain $3^2 - 5 \cdot 3 + 6 = 0$, or $9 - 15 + 6 = 0$, or $0 = 0$. Plainly the statement obtained is false when -2 is put for x, for then it becomes $(-2)^2 - 5(-2) + 6 = 0$, or $4 + 10 + 6 = 0$, or $20 = 0$. Similarly, any value other than 2 or 3, when put for x, gives a relation between numbers which is not true.

Every equation of condition may be regarded as asking a question. Thus the equation $3x + 2 = 15$ asks, "What number when multiplied by 3 and the product increased by 2 gives 15 as the result?"

The equations used in solving the problems on pages 46–48 are equations of condition. The conditions there expressed in ordinary language in the problems were translated into the algebraic language of equations.

Instead of the equality sign, the sign $\equiv$ (read *is identical with*, or *is identically equal to*) is sometimes used for emphasis if the expression is an identity.

Thus $3a = 2a + a$ may be written $3a \equiv 2a + a$.

32. Root of an equation. A number or literal expression which being substituted for the unknown letter in an equation reduces it to an identity is said to **satisfy** the equation.

Thus it has been shown that 4 satisfies the equation $4x = x + 12$, and, similarly, the literal expression $3a$ satisfies the equation $x - 5 = 3a - 5$.

A number or number symbol is called a ***root*** *of an equation if it satisfies the equation.*

The process of checking the solution is really finding out whether the result obtained is a root of the equation or not. It should be particularly noted that after the root has been substituted in place of the unknown, the equation of condition becomes an identity.

ORAL EXERCISES

In Exercises 1–10, state which expressions are identities and which are equations of condition:

1. $3 + 7 = 13 - 3.$
2. $2x = 8.$
3. $3a - 5 = 2a.$
4. $x + 1 = 1 + x.$
5. $x - 2 = 2x - 3.$
6. $2x = 4x - 3x + x.$
7. $x + 7 = 4 + x + 3.$
8. $3x - 1 + x + 1 - 4x = 0.$
9. $x + 4 - 3x + 2 = 6x - 8.$
10. $3x - 3 = 4x - 2 - x - 1.$

In Exercises 11–20, select from the numbers at the right of each equation those which satisfy that equation:

11. $4x - 12 = 0.$	1, 2, 3.
12. $2x = x + 2.$	1, 2, 4.
13. $x + 3 = 4x - 6.$	0, -1, 3.
14. $3x - 4 = 2x + 1.$	-2, 1, 5.
15. $x + 6 = 3x - 10.$	8, 5, -4.
16. $x + 1 - 2x + 3 = 0.$	7, 8, -3.
17. $x^2 + 3x + 2 = 0.$	-1, 1, 0.
18. $x^2 - 16 = 6x.$	8, -2, 3.
19. $4x^2 + 4x = 3.$	2, $\frac{1}{2}$, 3.
20. $5x^2 = 3x.$	1, 2, 0.

Is the number at the right of each of the following equations a root of that equation?

21. $3x - 24 = 0.$	8.	25. $x^2 - 16 = 4x + 5.$	7.
22. $4x - 15 = x.$	3.	26. $3x + 7 = 5x - 1.$	4.
23. $5x - 12 = 2x.$	2.	27. $y^2 + 5y + 4 = 0.$	2.
24. $x^2 + 6x - 3 = 0.$	4.	28. $z + 3 = 2z - 1.$	4.

33. Transposition. In solving the equation $5x - 4 = 18$ we add 4 to each member. If we indicate this addition, the equation becomes

$$5x - 4 + 4 = 18 + 4.$$

Now in the first member $-4+4=0$, so that these two numbers may be omitted, and the equation becomes

$$5x=18+4.$$

Upon comparing this with the original equation, it is seen that -4 has vanished from the first member of the original equation, and $+4$ has appeared in the second member of the new equation.

Again, if we subtract $3y$ from both members of the equation

$$6y=3y+12,$$

we get the equation $\quad 6y-3y=12,$

which differs from the original equation only in having $-3y$ in the first member instead of $+3y$ in the second.

It thus appears that a term may be omitted from one member of an equation, provided the same term with its sign changed from + to − or from − to + is written in the other member. This process is called **transposition.**

Hereafter, in order to simplify an equation, instead of subtracting a number from both members or adding a number to both members, as illustrated in the example on page 41, the student should use transposition, as it is usually more rapid and convenient. He should, however, always remember that the transposition of a term is really the subtraction of that term from each member of the equation.

Like terms in the same member of an equation should be combined before transposing any term.

NOTE. Our word *algebra* is derived from the Arabic word for *transposition.* The process by which one passes from the equation $px-q=x^2$ to the equation $px=x^2+q$ was known as *al-jebr.* This is the first word in the title of an Arabic book on algebra which was translated into Latin. For some reason only this part of the title remained, and by the early part of the seventeenth century *al-jebr,* or algebra, was the common name given to the whole subject.

ORAL EXERCISES

State the term or terms which must be added to both members of the following equations in order to transpose the underscored terms. What does each equation become after transposition?

1. $x^2 - \underline{5} = 4.$
2. $x + \underline{12} = 4.$
3. $x - 3 = \underline{2}.$
4. $x^2 + 6x = \underline{5x} - 3.$
5. $x^2 - 4x = \underline{5}.$
6. $2x^2 + 3x + 1 = \underline{2x - 4}.$
7. $2x + \underline{3} + 3x + \underline{5} = \underline{4x} + 6.$
8. $5x - \underline{2} - 3x + \underline{6} = 7 - \underline{x}.$
9. $x^2 + 4x = -\underline{3}.$
10. $3x - 2 = -\underline{x^2}.$
11. $5x^2 - 4x = \underline{3x} - \underline{1}.$
12. $8x - 2x^2 = \underline{4} - \underline{4x^2}.$

EXAMPLE

Solve the equation $5x - 7 + 2x + 6 = 11x - 5 - 6x + 12.$

Solution. $5x - 7 + 2x + 6 = 11x - 5 - 6x + 12.$

Combining like terms, $7x - 1 = 5x + 7.$

Transposing, $7x - 5x = 7 + 1.$

Combining like terms, $2x = 8.$

Dividing by 2, $x = 4.$

Check. $5x - 7 + 2x + 6 = 11x - 5 - 6x + 12.$

Substituting 4 for x, $20 - 7 + 8 + 6 = 44 - 5 - 24 + 12.$

Combining, $27 = 27.$

EXERCISES

Solve the following equations, and check results:

1. $4x - 1 = 3x + 3.$
2. $4x + 5 = x - 10.$
3. $-6y - 2 = 1 - 9y.$
4. $6x + 3 - 2x = 27.$
5. $4y + 3 = 2 + 5y.$
6. $4k + 1 - 2k = 4.$
7. $16 + 4x - 4 = x + 12.$
8. $4x - 15 = 15 - x - 5.$
9. $5n + 11 - 2n = 6 - 4.$
10. $2x - 1 = 29 + 7x.$

11. $3k + 9 + 5k - 33 = 0.$

12. $3h - 20 + 8h - 24 = 0.$

13. $6x - 13 + 2x + 3 = 0.$

14. $x + 2 - 5x = -9x + 12.$

15. $4x + 3 = 15 + 2x + 6.$

16. $3y + 5 + y + 3 = 0.$

17. $9 - 8h + 2 = 3 - 4h.$

18. $3 - 5x + 2 = 5 + 7x.$

19. $8x - 3x = 15x + 4 - 13x.$

20. $8 + 7y - 13 = y - 27 - 5y.$

21. $3x - 15 - 10x - 9 + 16x - 21 = 0.$

22. $18 + 5x - 6 - 2x + 1 + 3x - 25 = 0.$

23. $0 = 18 - 4x + 27 + 9x - 3 + 16x.$

24. $5n - 8 + 4n + 5 = 7n - 3 - 2n + 5.$

ORAL EXERCISES

Represent a number:

1. Greater by 4 than x.

2. Greater by a than x.

3. Less by 5 than n.

4. Less by b than n.

5. Three times y.

6. Four greater than $a + b$.

7. x greater than $a + b$.

8. Seven less than $x - 2y$.

9. a less than $2c - b$.

10. c less than three times x.

11. a greater than four times n.

12. Three less than five times y.

13. Eight greater than three times n.

14. One part of 10 is 4. What is the other part?

15. One part of x is 5. What is the other part?

16. One part of 8 is y. What is the other part?

17. One part of n is p. What is the other part?

18. One part of x is a. What is the other part?

19. One part of $x + y$ is z. What is the other part?

20. The sum of two numbers is 12. If one of them is 5, what is the other?

21. The sum of two numbers is 12. If one of them is p, what is the other?

22. The difference of two numbers is 8. If the greater is 25, what is the other?

23. The difference of two numbers is 8. If the greater is x, what is the other?

24. The sum of two numbers is 25. If one of them is a, what is the other?

25. The sum of two numbers is s. If one of them is 9, what is the other?

26. The sum of two numbers is a. If one of them is x, what is the other?

27. The difference of two numbers is 6. The less number is 4. What is the other?

28. The difference of two numbers is d. If the less of them is 4, what is the other?

29. The difference of two numbers is d. If the less of them is n, what is the other?

30. What is the excess of 19 over 6? 9 over n?

31. By how much does 24 exceed 16? 24 exceed x? y exceed 8? $a+b$ exceed 12?

32. How much greater is 45 than 18? than 33? than a? How much greater is x than y?

33. How much less is 15 than 32? than a? How much less is x than y?

34. By how much does $a+6$ exceed a? $a+6$ exceed b? $4x-3$ exceed $3x$?

34. Translation of problems into equations. In the solution of problems the writing of the equation is nothing more than translating from ordinary speech into

the language of algebra. Sometimes it is possible to translate the statement of the problem, word by word, into algebraic symbols.

For example,

Four times a certain number, diminished by 6,
$$4 \quad \times \quad n \quad - \quad 6$$
gives the same result as the number increased by 30.
$$= \quad n \quad + \quad 30$$

PROBLEMS

1. What number increased by 16 is equal to 37?

HINT. $n + 16 = 37$

2. What number diminished by 23 is equal to 9?

3. To what number must 28 be added so that the result may be 15?

4. From what number must 15 be subtracted so that the result may be 24?

5. If a certain number is doubled and the result diminished by 13, the remainder is 49. What is the number?

6. If a certain number is doubled and the result increased by 16, the sum is 12. What is the number?

7. If a certain number is trebled and the result diminished by 16, the remainder is equal to the original number. What is the number?

8. Three times a certain number, less 23, equals twice the number, less 8. Find the number.

9. Five times a certain number, increased by 12, equals eight times the number, diminished by 6. Find the number.

10. Four times a certain number, less 7, equals twice the number, plus 21. Find the number.

11. What number is as much less than 90 as it is greater than 16?

12. What number exceeds 16 by twice as much as 52 exceeds the number?

13. A certain number added to 14 gives the same result as that obtained when this number is subtracted from 58. What is the number?

14. If 9 is added to twice a certain number, and 12 is subtracted from five times the number, the results are the same. Find the number.

15. Five times a certain number, plus 16, equals three times the number, plus 10. What is the number?

16. The sum of two numbers is 62, and their difference is 10. Find the numbers.

Solution. The greater number + the less number = 62.

There are two unknowns in this problem, but both can be represented in terms of the same letter, thus:

Let $n =$ the less number.

Then $n + 10 =$ the greater number, since the smaller is 10 less than the greater.

Placing these symbols in the principal statement, we have

$$n + 10 + n = 62.$$

Combining, $2n + 10 = 62.$

Transposing, $2n = 62 - 10.$

Combining, $2n = 52.$

Dividing by 2, $n = 26$, the less number,

and $n + 10 = 36$, the greater number.

Check. $36 + 26 = 62,\ 36 - 26 = 10.$

17. The sum of two numbers is 58, and their difference is 12. Find the numbers.

18. The sum of two numbers is 8, and their difference is 42. Find the numbers.

19. The sum of two numbers is 21, and the second is 5 greater than the first. Find the numbers.

20. The sum of two numbers is 85, and one exceeds the other by 41. Find the numbers.

21. The sum of three numbers is 55. The second is 5 greater than the first, and the third is 17 greater than the first. Find the numbers.

22. The sum of three numbers is 16. The second is 10 less than the first, and the third is 2 less than twice the first. What are the numbers?

23. Find three consecutive integers whose sum is 90.

24. Find three consecutive odd integers whose sum is 33.

25. A rectangle whose perimeter is 32 feet is three times as long as it is wide. Find its dimensions.

26. A rectangle whose perimeter is 102 feet is twice as long as it is wide. Find its dimensions.

27. The length of a rectangle is 7 feet more than twice the width. Its perimeter is 110 feet. What are its dimensions?

28. A lawyer can afford to spend only $40 per week for office help. He must pay the office boy $5; and he estimates that the stenographer's work is worth two thirds as much as the bookkeeper's. How much should he pay each?

29. Mr. Brown can allow his three children all together a dollar and a half per week for spending money. In order to pay his car fare to school John needs 50 cents per week more than Elizabeth, while James requires only half as much as Elizabeth. What allowance will each child receive?

30. A farmer offers his son a reward if he will shingle the roof of the barn in 6 working days. The roof contains 120 courses of shingles. The son wishes to plan the work so that he may lay each day two courses less than the previous day. How many courses must he lay the first day?

CHAPTER VII

PARENTHESES

35. Removal of parentheses. In solving exercises and problems it is often necessary to inclose several terms in a parenthesis. Sometimes it is necessary to inclose this parenthesis with other terms in a second parenthesis or even in a third. To avoid confusing the different parentheses, *brackets* [ ] and *braces* { } are also used.

The parenthesis, the brackets, and the braces are called *signs of aggregation.* For convenience, brackets and braces are often spoken of as parentheses.

Occasionally the vinculum, or bar, is used in the same way as a parenthesis. Thus $-\overline{m+2a}=-(m+2a)$.

In the solution of equations and in other algebraic work it is frequently necessary to remove all signs of aggregation. This removal, while it depends only on the principles of addition and subtraction which have already been learned, requires, nevertheless, some special study to acquire speed and accuracy.

The value of $12+(5-3)$ is the same as that of $12+5-3$, or 14. Similarly, $a+(b-c)=a+b-c$.

The plus signs preceding the parentheses in $12+(5-3)$ and $a+(b-c)$ belong to these parentheses respectively and vanish with them, whereas the plus signs *understood* before 5 and b within the parentheses are supplied when we write $12+(5-3)=12+5-3$ and $a+(b-c)=a+b-c$. In the expression $12+(-5-3)$ the sign of 5 must be retained, and we have $12+(-5-3)=12-5-3=4$.

Therefore we have the

Principle. *A parenthesis and the sign before it, if plus, may be removed from an expression without changing the signs of the terms which were inclosed by the parenthesis.*

In the expression $12-(5-3)$ the sign before the binomial shows that $(5-3)$ is to be subtracted from 12. Hence we change the signs of the terms in the subtrahend and find the sum of the resulting terms and the minuend.

Thus $12-(5-3)=12-5+3=10$. This is obviously correct, for $12-(5-3)=12-2=10$.

Similarly, $a-(b-c)$ becomes $a-b+c$ when the signs of the subtrahend, $(b-c)$, are changed and the result is added to a.

The minus signs preceding the parentheses in $12-(5-3)$ and $a-(b-c)$ vanish with the parentheses, and the plus signs understood before 5 and b within the parentheses are changed when we write $12-(5-3)=12-5+3$ and $a-(b-c)=a-b+c$

Therefore we have the

Principle. *A parenthesis and the sign before it, if minus, may be removed from an expression, provided the sign of each term which was inclosed by the parenthesis be changed.*

These principles may also be applied to remove the parentheses used to inclose the numbers in Chapter II.

When one parenthesis incloses another, either the outer or the inner parenthesis may be removed first. It is best for the beginner to use the

Rule. *Rewrite the expression, omitting the innermost parenthesis, changing the signs of the terms which it inclosed if the sign preceding it be minus and leaving them unchanged if it be plus.*

Combine like terms that may occur within the new innermost parenthesis.

Repeat these processes until all the parentheses are removed.

EXAMPLE

Remove the parentheses from $8-(3-2a)+(4-5a)$.

Solution. $8-(3-2a)+(4-5a)=8-3+2a+4-5a=9-3a$.

The student should observe that the removal of a parenthesis is merely the performance of an *indicated* addition or subtraction. Thus $3-2a$ in the example above is a subtrahend which is to be taken from $8+4-5a$ even though it comes before $4-5a$.

The student should exercise great care in the removal of a parenthesis preceded by a minus sign. Errors in this connection are common and they persist long.

ORAL EXERCISES

Read the following expressions after removing the parentheses:

1. $8+(4+2)$. **5.** $x+(y+z)$. **9.** $a-(a-b)$.

2. $8+(4-2)$. **6.** $x+(y-z)$. **10.** $a+(-a+b)$.

3. $8-(4+2)$. **7.** $x-(y+z)$. **11.** $x+y+(x-y)$.

4. $8-(4-2)$. **8.** $x-(y-z)$. **12.** $a-b-(b-a)$.

13. $a-3-(3-a+x)$. **17.** $-(3a-c)-(2c-5a)$.

14. $(a-c)-(c+a)$. **18.** $x-(2a-3x-2c)$.

15. $-(2a-b)+(2b-a)$. **19.** $(-x+a)-(a-3x)$.

16. $-(a-3c)+(2-x)$. **20.** $-(-a+x)+(-a-5x)$.

EXERCISES

Remove the parentheses and combine like terms:

1. $14-(6-3)-5$. **3.** $(7-5+2)-(6-4)+12$.

2. $10+(7-4)-(9-7)$. **4.** $11a-(4a-9a)+(6a-a)$.

5. $(2b-5a)-(4a-b-7a)$.

6. $a-(b-a)+(2b-3c)$.

7. $a-b-(c-d)+(a-b)-(b-c)$.

8. $(x-y)-(2y-3x)+(x-4y)$.

9. $x-(x-y-2z)-(3z+y+4)+(x-6)$.

10. $7-[8-(3-10)]-(13-25)$.

HINT. $7-[8-3+10]-13+25$, etc.

Observe that we may first remove the *inner* parenthesis, $-(3-10)$, and also remove the last parentheses at the same time. Doing this often saves rewriting and decreases the probability of error.

11. $8-[5-(4-6)]-(6-15)$.

12. $12-(8-17)-[(4-1)-8]$.

13. $16-[3-(2-5)]+(3-5)-(8-4)$.

14. $a+[2a-(3a-2b)]+(3b-2a)$.

15. $a+[5a-(3x-2a)]-(4a-3x)$.

16. $2a-[6a-(5x-4a)]+(8a-7x)$.

17. How many parentheses may be removed the first time Exercise 18 is rewritten? Exercise 19? 20? 21? 22? 23? 24? 25?

18. $(5x-6y)-[-4x-(4z-y)-2z]$.

19. $[3x-(2y+z)]-[-(3y-2x)+5x]$.

20. $[(a+3)-(x-5)]-[a+3+(x-5)]$.

21. $7-[-6-\{-4+(9-10)\}+11]$.

22. $5a-[2a+(-3a+4b)-(a-8b)+4a]$.

23. $2x-3y-[\{3z-7x-(y-4z)-9x\}+z]$.

24. $\{4a-[2a-(8a+2b)+4]-(4b-6)\}$.

25. $-5x+[15x-\{11x-(2x-7x-4)-3x\}-22]$.

26. $(4y-7x)-\{3x-[4x-(7y-4x)-(-5y+3x)]\}$.

Sometimes it is necessary to remove some of the signs of aggregation in an expression, leaving others. In the following remove the parentheses, retaining the brackets, and simplify the results as much as possible:

27. $[(a+b)+c]$, $[(a+b)-c]$.

28. $[4x+(3z-5y)]$, $[4x-(3z-5y)]$.

29. $[(a-b)+(b-2a)]$, $[(a-b)-(b-2a)]$.

30. $[(3a-2b)+(2b-3a)]$, $[(3a-2b)-(2b-3a)]$.

31. $[(a-2b)+(3c-d)]$, $[(a-2b)-(3c-d)]$.

32. $[(4x-3)+(5y-7)]$, $[(4x-3)-(5y-7)]$.

33. $[(x^2-a^2)+(y^2-2a^2)]$, $[(x^2-a^2)-(y^2-2a^2)]$.

36. Inclosing terms in parentheses. Obviously,

$$16+9-5=16+(9-5), \text{ for each equals } 20.$$

Similarly, $$a+b-c=a+(b-c).$$

That the two preceding expressions are equal may be seen by removing the parentheses according to the first Principle on page 65.

These processes are illustrations of the

Principle. *One or more terms may be inclosed in a parenthesis preceded by a plus sign, without changing the sign of any of the terms.*

The expression

$$17+8-3=17-(-8+3), \text{ for each equals } 22.$$

Similarly, $$a+b-c=a-(-b+c),$$

and $$a-b+c=a-(b-c).$$

That the right member in each of these cases is another form of the left may be seen by removing the parentheses according to the second Principle on page 65.

These processes are illustrations of the

Principle. *One or more terms may be inclosed in a parenthesis preceded by a minus sign, provided the sign of each term thus inclosed is changed.*

EXERCISES

In the following, inclose in a parenthesis preceded by a plus sign all the terms containing the letters x or y, and inclose in a parenthesis preceded by a minus sign all the other terms:

1. $x^2 - a^2 - 2a$.

2. $12a + x^2 - 9 - 4a^2$.

3. $y^2 - 9b^2 + 6ab - a^2$.

4. $10ab + x^2 - a^2 - 25b^2$.

5. $x^2 - b^2 - 4b^2 + y^2 - a^2 - 2xy$.

6. $-4ab + x^2 - 4b^2 + y^2 - a^2 - 2xy$.

7. $16x^2 - a^2 - 16xy - 4 + 4a + 4y^2$.

8. $x^2 - b^2 - 10xy + 12ab - 36a^2 + 25y^2$.

9. $2ab - a^2 + x^2 + 2xy - b^2 + y^2$.

10. $x^2 - 16xy - a^2 + 16a + 64y^2 - 64$.

11. In Exercises 1–4, inclose the last two terms in a parenthesis preceded by a negative sign.

12. In Exercises 5–10, inclose the last three terms in a parenthesis preceded by a negative sign.

CHAPTER VIII

MULTIPLICATION

37. Product of terms containing unlike letters. The student is familiar with the fact that the factors of a product may be written in any order.

For example, $2 \cdot 4 = 4 \cdot 2.$

Similarly, $a \cdot b = b \cdot a.$

This principle is called the Commutative Law of Multiplication.

Further, $2a^2 \times 3 = 2 \times 3 \times a^2 = 6a^2,$

and $2a \times 3b = 2 \times 3 \times a \times b = 6ab.$

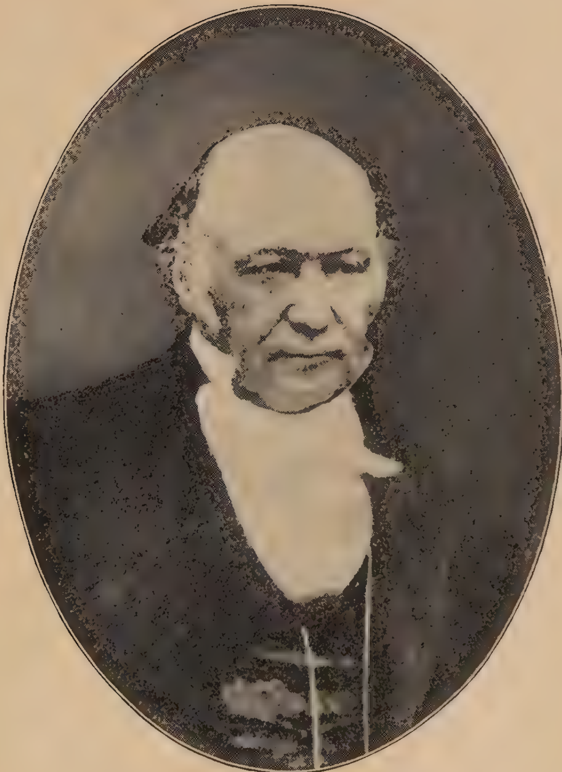
Similarly, $6x^2 \cdot 5y^3 = 6 \cdot 5 \cdot x^2 \cdot y^3 = 30x^2y^3.$

Also, $4ab \cdot 3z^2 = 4 \cdot 3 \cdot ab \cdot z^2 = 12abz^2.$

Up to the present we have assumed that the various operations of multiplication in any product may be performed in any order.

That is, $(3 \cdot 2)4 = 3(2 \cdot 4) = 24$. In general terms $a(b \cdot c) = (a \cdot b)c$. This merely tells us that a multiplied by the product of b and c is the same as the product of a and b multiplied by c. This principle is called the Associative Law of Multiplication.

Biographical Note. *Sir William Rowan Hamilton.* It is strange that of all the topics treated in this book the last to be thoroughly understood by mathematicians are those appearing in the first chapters. But in all the sciences it is often most difficult to answer the questions that at first sight seem quite obvious. Any child can ask what electricity is, but the wisest scientist cannot tell. He can only explain what electricity



SIR WILLIAM ROWAN HAMILTON

does. It is easy to ask how the earth came to be revolving around the sun with the moon revolving around it, but even the deepest students of astronomy differ in their theories of how it came to be. And so in mathematics, long after many of the more complicated processes of algebra were completely understood, the simple laws of operation of numbers were surrounded with haze. One of the men who did most to clarify the nature of these laws was Sir William Rowan Hamilton (1805–1865). He was born in Dublin, Ireland, where he lived most of his life. He was a precocious boy, and at the age of twelve was familiar with thirteen languages. He devised kinds of numbers that do not follow the same laws as those that we use in algebra, and so threw a flood of light on the nature and properties of these common numbers. He was the first to recognize the importance of the Associative Law, and called it by that name. Most of his works are very advanced in character and are difficult to read.

38. Product of terms containing like letters. By the definition of an exponent (p. 9), $a^2 = a \cdot a$, and $a^3 = a \cdot a \cdot a$.

Therefore $a^2 \times a^3 = a \cdot a \times a \cdot a \cdot a = a^5 = a^{2+3}$.

Similarly, $b \times b^3 \times b^5 = b \times b \cdot b \cdot b \times b \cdot b \cdot b \cdot b \cdot b = b^9 = b^{1+3+5}$.

In like manner $3^2 \times 3^4 \times 3^5 = 3 \cdot 3 \times 3 \cdot 3 \cdot 3 \cdot 3 \times 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^{11}$
$= 3^{2+4+5}$.

Also, $ay^2 \times y^3 = ay^5 = ay^{2+3}$,

and $2\,ab \times 3\,a^2 = 6\,a^3b = 6\,a^{1+2}b$,

and $4\,x^2yz \times 5\,xy^3 = 20\,x^3y^4z = 20\,x^{2+1}y^{1+3}z$.

Therefore we have the

Principle. *The exponent of any letter in the product is equal to the sum of the exponents of that letter in the factors.*

This is expressed in general terms, thus:

$$n^a \times n^b = n^{a+b}.$$

The law of signs for the multiplication of positive and negative numbers, given on page 27, applies to literal terms as well.

Thus
$$+2\,a^2 \times (+3\,a^5) = +6\,a^7.$$
$$+2\,a^2 \times (-3\,a^5) = -6\,a^7.$$
$$-2\,a^2 \times (+3\,a^5) = -6\,a^7.$$
$$-2\,a^2 \times (-3\,a^5) = +6\,a^7.$$

For the multiplication of two monomials we have the

Rule. *Obeying the rule of signs for multiplication, write the product of the numerical coefficients followed by all the letters that occur in the multiplier and the multiplicand, each letter having as its exponent the sum of the exponents of that letter in the multiplier and the multiplicand.*

ORAL EXERCISES

Perform the following indicated multiplications:

1. $(3)(-6)$.
2. $(-2)(7)$.
3. $(-8)(-3)$.
4. $(-4x)(5)$.
5. $3x \cdot 5x$.
6. $x^2 \cdot x^3$.
7. $x^2 \cdot x^5$.
8. $x^4 \cdot x^5$.
9. $x^2 \cdot x^2$.
10. $x^3 \cdot x^3$.
11. $x^5 \cdot x^7$.
12. $-x^2 \cdot (x^4)$.
13. $-x^3 \cdot (-x^2)$.
14. $-x^2 \cdot (-3x^3)$.
15. $-x^2 \cdot (2x^5)$.
16. $(-2x^2)(3x^3)$.
17. $(3x^4)(-2x^5)$.
18. $(4x^2)(-5x^4)$.
19. $(-3x^3)(-4x^4)$.
20. $(-6x^2)(4x)$.
21. $(-7x)(-2x^3)$.
22. $(+2a^2)(-3a^2x)$.
23. $(-4a^3x)(+2x^4)$.
24. $(-5a^2x)(-7ax)$
25. $(+3a^2x^3)(4a^2x^2)$.
26. $(-4x)^2$.
27. $(7)(-5a)$.
28. $(3a)(-8)$.
29. $(-2y)^2$.
30. $(-9a)(-10)$.
31. $(-3ax)^2$.
32. $(4a)(-5a)$.
33. $(6abc)^2$.
34. $(-11x)(3x)$
35. $(7x)(-3x)$.
36. $(-a)^3$.
37. $(-2a)^3$.
38. $(3a^2x)^2$.
39. $(-2a)(-4a^2)$.
40. $(5a^4)(8a^3)$.
41. $(-5x)^3$.
42. $(a^8)(-16a)$.
43. $(-4a^5)(-6a^3)$.
44. $(+3y)^3$.
45. $(4x)(5y)(6xy)$.
46. $(3x)(-2y)^2$.
47. $(-3a^2x)^2$.
48. $(3x^2)(-y)$.
49. $(5x^2y)(-2x^3)$.
50. $(-6x^3y^2)^2$.
51. $(-x^4y)(-x^2y^4)$.
52. $(2ax^2)^3$.
53. $(5a^3)(-4a^2)(-3a)$.
54. $(-2a^2)(3a^3)^2$.
55. $(3ax)(-2a^2x)(-7ax^3)$.

39. Multiplication of a polynomial by a monomial. Clearly $2(5+3)$ is equivalent to $2 \cdot 5 + 2 \cdot 3$, each expression being equal to 16.

Similarly, $a(b+c) = ab + ac$. This principle is called the Distributive Law of Multiplication.

Therefore, for the multiplication of a polynomial by a monomial, we have the

Rule. *Multiply each term of the polynomial by the monomial and write in succession the resulting terms with their proper signs.*

Example. $3x^2 - 2xy + 4y - 5a - 6$

$2xy$

Product, $6x^3y - 4x^2y^2 + 8xy^2 - 10axy - 12xy$

EXERCISES

Multiply:

1. $x + 2$ by 2.

2. $x - 4$ by x.

3. $x^2 + 5$ by $2x$.

4. $3x^2 + 4$ by $2x$.

5. $x^2 - 2x$ by x^2.

6. $y^2 - 3y + 2$ by $3y$.

7. $y^3 - 5y^2 + 3y - 1$ by y^2.

8. $5x^2 - 2x - 4$ by x^2.

9. $-4x^2 + 6x - 5$ by $6x^2$.

10. $2x^2 - 3x - 2$ by $-4x^2$.

11. $x^3 - 3x^2 - 4$ by $-5x^4$.

12. $2x - 3x^2 - 2x^3$ by $-2x^3$.

13. $7xy - x + y$ by $3xy$.

14. $x^2 - 2xy + y^2$ by $-3xy$

15. $a^4 - a^2b^2 + b^4$ by $-a^2b^2$.

16. $-a^2x^2 - 2ax + 7b^2$ by $-4abx$.

17. $7x^3 - 8x^2 + 12x - 6$ by $-4x^3$.

18. $-9a^2 - 12ax + 42x^2$ by $3ax^3$.

Perform the multiplication indicated:

19. $7(2x - 3)$.

20. $5x(x - y)$.

21. $-8(3x - 7)$.

22. $-9(-4a + b)$.

23. $-3x(2x - 7)$.

24. $6x^2(9x^3 - 4x)$.

25. $-3(x^2-2x-6)$.

26. $5xy(x^2-6x+9)$.

27. $-3x(ax-bx-3cx^2)$.

28. $(x^3-ax+a^2)(-2a^2x)$.

29. $-7ab(ax^2-bx+c)$.

30. $4x^2(-3x+7x^2-x^4)$.

40. Multiplication of polynomials. Clearly $(5+3)(7-4) = 8\cdot 3 = 24$. The multiplication may also be performed as follows: $(5+3)(7-4) = 5(7-4)+3(7-4) = 35 - 20 + 21 - 12 = 24$.

Similarly, $(2x+3)(4x-5) = 2x(4x-5)+3(4x-5) = 8x^2-10x+12x-15$, or $8x^2+2x-15$.

In general terms $(a+b)(c+d) = a(c+d)+b(c+d) = ac+ad+bc+bd$.

This gives for the multiplication of polynomials the

Rule. *Multiply the multiplicand by each term of the multiplier in turn, and add the partial products.*

Example.

	$3x - 2$	
	$2x + 3$	
Multiplying by $2x$,	$6x^2 - 4x$	= first partial product.
Multiplying by $+3$,	$+9x - 6$	= second partial product.
Complete product,	$6x^2 + 5x - 6$	= sum of partial products.

EXERCISES

Multiply:

1. $x+4$ by $x+3$.
2. $2x+3$ by $x+3$.
3. $4x+7$ by $3x+2$.
4. $3x-5$ by $3x+8$.
5. $3x-2$ by $2x+3$.
6. $6-4a$ by $5a-7$.
7. $2x+y$ by $x+3y$.
8. $2x-3y$ by $3x-2y$.
9. $3x-2$ by $2x-3$.
10. $-3x+11a$ by $5x-a$.
11. $ax-bx$ by $cx+dx$.
12. $-cx+d$ by $bx-cx^2$.
13. $4x-3y^2$ by $6x+5y$.
14. x^2-5x+6 by $x-3$.
15. $3x^2-3x-7$ by $2x+4$.
16. x^2-xy+y^2 by $x+y$.

17. $a^2x^2 - 2a^2x + 4a^2$ by $ax + 2a$.

18. $3x^3 - x^2 - 5x$ by $2x^3 - 5x^2$.

19. $2x^2 - 7x + 12$ by $x^2 - 3x - 5$.

20. $x^2 - 2x - 3$ by $x^2 - 2x - 3$.

Expand:

21. $(3x^2 - 5x - 1)(2x - 3)$.

22. $(x^2 - 3x - 2)(x^2 - 2x + 3)$.

23. $(t^2 - 3t + 2)^2$.

24. $(3y - y^2 - 2)^2$.

25. $(4x - 2x^2 - 5)^2$.

26. $(3x^2 - 4x + 7)^2$.

41. Powers. A **power** of a number is the product obtained by using the number as a factor one or more times.

For example, 8, or 2^3, is the third power of 2; 81, or 3^4, is the fourth power of 3; and $32x^5$, or $(2x)^5$, is the fifth power of $2x$.

42. Arrangement. A polynomial is said to be **arranged** according to the *descending* powers of a certain letter when the exponents of that letter in successive terms decrease from left to right. Thus $2x^4 - 5x^2 - 6x + 8$ is arranged according to the descending powers of x. Again, $4 - 2y + y^2$ and $x^3 - 3x^2y + 3xy^2 - y^3$ are arranged according to the *ascending* powers of y.

Whenever it is possible to arrange the multiplier and the multiplicand in the same order with respect to the same letter it should be done, as the addition of the partial products is then much more easily performed.

43. Degree. The **degree** of a term with respect to any letter which does not appear in the denominator is determined by the exponent of that letter in the term.

Thus x, $3xy$, and $4a^2xz$ are of the first degree in x, and $3xy^2$ is of the second degree in y.

The degree of a term with respect to *two or more* letters which do not appear in the denominator is determined by the *sum of the exponents* of those letters in that term.

Thus $5x^3y$ is of the fourth degree in x and y; $4a^2bc^3x^2y$ is of the sixth degree in a, b, and c.

44. Check of multiplication. The work of multiplication can be checked by giving a small, convenient numerical value to each letter involved and finding the corresponding numerical values of the multiplier, the multiplicand, and the product. The product of the numerical values of the multiplier and the multiplicand should equal the numerical value of the product.

The least positive integer which gives a reliable check by the method outlined above is the number 2. This is true only when one letter is involved. If more than one letter is involved, the check is not certain if 2 is substituted for each letter.

The number 1 is very convenient to use in checking, but it will not check exponents, since $x^2 = x^3 = x^5 = x^{10}$, etc., if $x = 1$.

EXAMPLE

Multiply $3x^3 - 5 + x^2 - 2x$ by $x^2 - 6 - 5x$, and check result:

Solution and check. Arranging both multiplier and multiplicand in descending powers of x and multiplying, we obtain:

$$
\begin{array}{lllllllll}
 & 3x^3 + x^2 - 2x - 5 & & & & = 24 + 4 - 4 - 5 & = & 19 & (\text{If } x = 2) \\
 & \underline{x^2 - 5x - 6} & & & & = 4 - 10 - 6 & = & \underline{-\ 12} & \\
 & 3x^5 + & x^4 - & 2x^3 - & 5x^2 & & & -228 & \\
 & & -\ 15x^4 - & 5x^3 + & 10x^2 + 25x & & & & \\
 & & & -\ 18x^3 - & 6x^2 + 12x + 30 & & & & \\
\textit{Product,} & 3x^5 - & 14x^4 - & 25x^3 - & x^2 + 37x + 30 & & & & \\
\textit{Product,} & 96 & -224 & -200 & -\ 4 \ +74 \ +30 & & = & -228 & (\text{If } x = 2)
\end{array}
$$

Since -228 is obtained in both steps of the check, the result is correct.

ORAL EXERCISES

1. Arrange $3x^2 - 8x^3 - 7 - x^4 - 2x$ in descending order.

2. Arrange $a^4 + b^4 + 6a^2b^2 - 4ab^3 - 4a^3b$ in descending order with reference to the letter b; with reference to the letter a.

3. Why should multiplier and multiplicand be arranged in the same order before attempting to multiply?

4. What is the value of a^3b^2 and of a^2b^3 if $a = b = 2$?

5. What point in checking does the preceding question bring out?

6. If 1 or 0 is correctly substituted for the letter or letters in a check of multiplication and the two results do not agree, is the multiplication necessarily incorrect?

7. Why is a check of multiplication unreliable when the number 1 is substituted for the letter or letters involved?

8. If one interchanges the multiplier and multiplicand and performs the multiplication a second time, would this give a check on the first? Explain.

EXERCISES

Multiply, and check every fifth exercise:

1. $3x^2 - 5x - 2$ by $2x - 3$.
2. $3x^2 + 2x - 7$ by $2x^2 - 5x - 3$.
3. $x^2 + 2x - 4$ by $x^2 + 2x + 4$.
4. $3a^2 - 8a - 1$ by $a^2 + 2a - 3$.
5. $2x^2 - 7x - 2$ by itself.
6. $2x^3 - 3x + 2$ by itself.
7. $a^2 - \frac{1}{2}a + \frac{1}{4}$ by $a^2 - a + 1$.
8. $x^2 - xy + y^2$ by $x^2 + xy + y^2$.
9. $3x^3 + 5x^2 - x + 2$ by $x^2 - 2x + 3$.
10. $a - 3 - a^3 - 2a^2$ by $2a - 3a^2 - 1$.

HINT. Arrange multiplier and multiplicand in descending order first.

11. $3a - a^3 + 6$ by $4a - 3a^2 - 5$.

Expand:

12. $(x^3 - x - 5)(3x^2 - 2x - 4)$.
13. $(3x - x^3 + x^2 - 6)(5 - x^2 - 3x)$.
14. $[3x - 2a - (2a - 3x)][3x - 2a + (2a - 3x)]$.
15. $(4a - 5a^2 + 7 + a^3)(3 + a^3 - a + a^2)$.
16. $(5x - 3 + 8x^3)(8 - 5x^2 + 2x^3 - 9x)$.
17. $(x^2 - 2xy + 3y^2)(x^2 + 2xy + 3y^2)$.
18. $(x^2y - y^2x)(4xy - 5x^2y)(3x^2y - 2xy^2)$.
19. $(x^2 + y^2 + z^2 - xy - xz - yz)(x + y + z)$.
20. $(a + b + c)^2$.
21. $(2a - 3b + 4c)^2$.
22. $(c + d - \frac{1}{2})^2$.
23. $(2a - 4b + 3)^2$.
24. $(a - 2b + 3c - 4d)^2$.
25. $(x + y + z)^3$.

Expand and collect like terms in:

26. $(x + 2y)^2 - (x - 2y)^2$.
27. $(x - 3)^3 + (3 - x)^3$.
28. $(x + y)^3 + (x - y)^3$.
29. $(2x - 3a)^3 - (3a - 2x)^3$.
30. $(ax - 5ay)^3 - (ax - ay)^3$.
31. $(4x - 3y)^2 - (2x + 5y)^2$.
32. $(x - 3)^3 - (2x - 1)^2$.
33. $(2x^2 - 3a)(2x^2 + 3a)^2$.
34. $(x^3 - 5)(x^3 + 4) - (x^5 + x^2)$.
35. $[a + (a^2 + 3)][a - (a^2 + 3)]$.
36. $[a - (x^2 - 3)][a + (x^2 - 3)]$.
37. $[x - a + (x^2 - 2)][(x - a) - (x^2 - 2)]$.
38. $(2x - 3)(3x^2 - 5x) - (4x^2 - 3x)(2x - 7) - (3x - 6)^2$.

CHAPTER IX

PARENTHESES IN EQUATIONS

45. Simple equations involving parentheses. In handling parentheses it is very easy to acquire careless habits, which are difficult to overcome. Accuracy in such work demands especial care in removing each parenthesis that is preceded by a minus sign.

EXAMPLE

Solve the equation $3(2x + 1) - (4x - 7) = 16$.

Solution.	$3(2x + 1) - (4x - 7) = 16.$
Removing parentheses,	$6x + 3 - 4x + 7 = 16.$
Combining,	$2x + 10 = 16.$
Transposing,	$2x = 6.$
Dividing by 2,	$x = 3.$
Check.	$3(2 \cdot 3 + 1) - (4 \cdot 3 - 7) = 16.$
Simplifying,	$21 - 5 = 16,$
or	$16 = 16.$

EXERCISES

Solve and check :

1. $2(x + 3) = 12.$

2. $5(x - 1) = 30.$

3. $4(x + 6) = 16.$

4. $2(3 - x) + 1 = 2.$

5. $5(x - 3) + 14 = 4.$

6. $5(x - 7) + 8x = 4.$

7. $4(4x - 1) + 3 = x.$

8. $16 + 2(4y - 7) - 12y = 0.$

9. $5x - 4(4 - x) - 11 = 0.$

10. $2(x + 1) - 3 = 3(x - 1).$

11. $4(2x - 5) + 15 = 3(x + 10).$

12. $3(x + 6) + 8 = 5(6 + x).$

13. $7(x + 5) = 4(x + 8) + 3.$

14. $7(y - 2) - 2(3 + y) = 0.$

15. $9y - 3(2y - 4) = 2(5 - y) + 7.$

16. $2n - 9(2n + 4) = 2(n - 9).$

17. $7x + 12 - 2(x - 5) = x - 14.$

18. $5(3h - 1) - 7h = 3(h + 7) - 1.$

19. $(n - 4)(n + 8) = 7 - (3 - n)(n + 5).$

Solution. Expanding, $(n^2 + 4n - 32) = 7 - (15 - 2n - n^2).$

Removing parentheses, $n^2 + 4n - 32 = 7 - 15 + 2n + n^2.$

Subtracting n^2 from each member, and combining,

$$4n - 32 = -8 + 2n.$$

Transposing and combining, $2n = 24.$

Dividing by 2, $n = 12.$

Check. $(12 - 4)(12 + 8) = 7 - (3 - 12)(12 + 5).$

Simplifying, $8 \cdot 20 = 7 - (-9 \cdot 17);$

that is, $160 = 7 - (-153),$

or $160 = 160.$

20. $(x + 3)^2 - (x + 5)^2 = -40.$

21. $(x + 2)^2 - (x - 4)^2 + 48 = 0.$

22. $(2x - 4)(3x - 6) = 6x^2 + 72.$

23. $(x + 4)(x + 6) = (x + 18)(x + 13).$

24. $(h + 2)(h + 3) = (h - 5)(h - 2).$

25. $(k - 7)(5 + k) - (k - 5)(k + 7) + 5 = 0.$

ORAL EXERCISES

1. The length of a rectangle is $x - 3$ and its width is 4 What is its area? its perimeter?

2. The length of a rectangle is a and its breadth is b. What is its area? its perimeter?

3. Express as a product the area of a rectangle whose length is $3x + 2$ and whose breadth is $x - 1$. What is its perimeter?

4. Each of three sheep cost \$40. What was the cost of all?

5. Each of n horses cost \$200. What represents the cost of all?

6. Each of a books cost b cents. What represents the cost of all?

7. What is the total cost of x hats at a dollars each, and y hats at b dollars each?

8. What is the cost of h horses at $x + 15$ dollars each?

9. Represent the total cost of a chairs at $y + 4$ dollars each, and b chairs at $z - 2$ dollars each.

10. A is n years old. What will be his age 6 years hence? x years hence? What was his age 5 years ago? a years ago?

11. A's age in years is three times B's. If B is x years old, represent A's age (a) now, (b) 7 years hence, (c) 4 years ago, (d) the sum of their ages 4 years hence.

12. A's age is $2n - 3$ years. What will be his age 12 years from now? b years from now? What was his age 7 years ago? x years ago?

13. A and B each have d dollars. If A gives B five dollars, how much will each then have?

14. A and B each have $x + 25$ dollars. If B gives d dollars to A, how much will each then have?

15. If A has $x + 25$ dollars, and B has $2x + 8$ dollars, express as an equation each of the following statements:

(a) A has as many dollars as B.
(b) A and B together have $250.
(c) A has $15 less than B.
(d) If A gains $75 and B loses $20, they have equal amounts.

16. If A's age is n years, B's $3n + 5$ years, and C's $2n - 4$ years, express

(a) the ages of A, B, and C six years hence;
(b) the ages of A, B, and C five years ago.

Express each of the following statements as an equation:

(c) The sum of the ages of A and B six years hence will be 50 years.

(*d*) The difference of the ages of C and A four years ago was 19 years.

(*e*) In 12 years A will be as old as B is now.

(*f*) Six years ago C was as old as A will be 15 years hence.

(*g*) In x years B will be 54 years.

(*h*) In five years the sum of the ages of A, B, and C will be 100 years.

17. A picture is 10 inches wide and 12 inches long and has a frame 2 inches wide. What are the outside dimensions of the frame?

18. If the frame in the preceding exercise were x inches wide, what would represent the outside dimensions of the frame? the area of the picture and frame? the area of the picture? the area of the frame?

46. Problems involving parentheses. The following problems involve two or more unknowns and the use of parentheses. One of the unknowns can always be represented by a single letter and the others by binomials involving this letter and one or more numbers. It will be necessary in some of the problems to inclose each of these binomials in a parenthesis and to think of them and use them as if each represented a single number. When the student can use a binomial in this way as readily as he uses a single letter, like x, he has made considerable progress in the algebraic way of thinking.

So far as possible the method of translating the problem into the symbolic language of algebra which was mentioned on page 61 should be followed.

For example: The sum of two numbers is 34. Four times the less equals three times the greater, plus 10. Find the numbers.

Here there are two unknowns, the greater number and the less. Each can be represented in terms of a letter, or this letter and a number, as follows:

Let n represent the less number. Then $34 - n$ must represent the greater.

By the conditions of the problem,

four times the less equals three times the greater, plus 10.

$$4 \times n = 3 \times (34 - n) + 10,$$

or $$4n = 3(34 - n) + 10.$$

Again: Seven times A's age two years ago equals five times his age ten years hence.

$$7 \times (a - 2) = 5 \times (a + 10),$$

or $$7(a - 2) = 5(a + 10).$$

PROBLEMS

1. The sum of two numbers is 49. Twice the greater equals 7 plus five times the less. Find each number.

2. The sum of two numbers is 45. Ten times the less equals 300 minus five times the greater. Find each number.

3. The sum of two numbers is 15. Twice one of them, minus ten times the other, equals zero. What are the numbers?

4. Separate 75 into two parts such that 29 plus four times the less equals three times the greater.

5. Twice a certain integer, plus four times the next consecutive integer, is 106. What are the integers?

6. Three times a certain integer, subtracted from four times the next consecutive integer, is 17. What are the integers?

7. The sum of two numbers is 15. Seven times one number equals 54 plus ten times the other. Find the numbers.

8. Twice a certain number equals 240 plus five times a second number. The sum of the numbers is 15. Find the numbers.

9. The perimeter of a rectangle is 90 feet, and five times the greater side plus four times the less equals 207. What are the sides?

10. The difference of the squares of two consecutive integers is 35. Find the integers.

11. The difference of the squares of two consecutive integers is 71. Find the integers.

12. The difference of the squares of two consecutive odd integers is 104. Find the integers.

13. The square of an integer plus the square of the next consecutive integer is 17 less than twice the square of the greater integer. Find the integers.

14. The difference of the squares of two consecutive odd integers is 48. Find the integers.

15. The product of two consecutive odd integers is 42 less than the square of the greater integer. Find the integers.

16. The product of two consecutive even integers equals 44 increased by the square of the smaller. Find the integers.

17. A's age in years is three times B's, and C is 10 years older than B. The sum of their ages is 45 years. Find the age of each.

18. A's age in years is twice B's, and C is 7 years older than A. Six years hence the sum of their ages will be 85 years. How old is each?

19. A is 10 years older than B, and C is 6 years younger than B. Six years ago the sum of their ages was 40 years. Find the age of each.

20. A is 2 years more than twice as old as B, and C is 7 years younger than A. In 6 years the sum of their ages will be 75 years. Find the age of each.

21. A is now 50 and B is 36 years old. How many years ago was A three times as old as B?

22. A is now 19 years old and B is 54. In how many years will A be exactly half as old as B?

23. A is 30 years older than B. In 20 years A will be twice as old as B. Find the age of each now.

24. A is three times as old as B. In 15 years A will be twice as old as B. Find the present age of each.

25. A's age is 8 years more than twice B's age. Sixteen years ago A was four times as old as B. Find the age of each now.

26. A square has the same area as a rectangle whose length is 8 inches greater, and whose breadth is 4 inches less, than the side of the square. Find the dimensions of each.

Solution. By the conditions of the problem,

the area of the square = the area of the rectangle.

Let s = the length of the side of the square in inches.

Then $s + 8$ = the length of the rectangle in inches,

and $s - 4$ = the breadth of the rectangle in inches.

Now the area of the square is $s \cdot s$, or s^2, square inches, and the area of the rectangle is $(s + 8)(s - 4)$, or $s^2 + 4s - 32$, square inches.

Therefore $$s^2 = s^2 + 4s - 32. \qquad (1)$$

Solving (1), $s = 8$, the length of the side of the square;

$s + 8 = 16$, the length of the rectangle,

and $s - 4 = 4$, the breadth of the rectangle.

Check. The area of the square is $8 \cdot 8 = 64$ square inches, and the area of the rectangle is also $16 \cdot 4 = 64$ square inches.

27. A square field has the same area as a rectangular field whose length is 15 rods greater, and whose breadth is 10 rods less, than the side of the square. Find the dimensions of each field.

28. A tennis court, for singles, is 3 feet shorter than three times its breadth. The distance around the court is 210 feet. Find the length and the breadth of the court.

29. A tennis court, for doubles, is 6 feet longer than twice its breadth. The perimeter of the court is 228 feet. Find the dimensions of the court.

30. The breadth of a basket-ball court is 20 feet less than its length. The perimeter of the court is 80 yards. Find the dimensions.

31. The perimeter of a rectangular athletic field is 780 feet. Its length is 5 yards less than twice its breadth. Find the dimensions.

32. In a rectangle 24 feet broad and 30 feet long a grass-plot is to be laid out, surrounded by a flower bed of uniform width. It is desired that the perimeter of the grassplot be exactly one half that of the entire rectangle. How wide should the flower bed be made?

33. The value of 15 pieces of money, consisting of nickels and dimes, is 90 cents. Find the number of each.

Solution. By the conditions of the problem,

the value of the dimes + the value of the nickels = 90 cents.

Let d = the number of dimes.

Then $15 - d$ = the number of nickels.

Now $10d$ = the value of the dimes in cents,

and $5(15 - d)$ = the value of the nickels in cents.

Therefore $10d + 5(15 - d) = 90. \quad (1)$

Solving (1), $d = 3$, the number of dimes,

and $15 - d = 12$, the number of nickels.

Check. $3 \cdot 10 + 12 \cdot 5 = 30 + 60 = 90.$

34. The value of 35 coins, consisting of dimes and quarters, is \$6.50. Find the number of each.

35. The value of 30 coins, consisting of nickels and dimes, is \$2.60. Find the number of each.

36. A collection of nickels, dimes, and quarters amounts to \$10.80. There are 5 more nickels than dimes, and the number of quarters is double the number of nickels and dimes together. Find the number of each.

CHAPTER X

DIVISION

47. Division of monomials. The rule for division of numerical terms was stated on page 29.

Just as $2 \div 3$ is written $\frac{2}{3}$, so $a \div b$ may be written as a fraction, $\frac{a}{b}$.

Similarly, $$a^2 \div x^2 = \frac{a^2}{x^2},$$

and $$2a \div 3b = \frac{2a}{3b}.$$

But $$12c^2 \div 4b^2 = \frac{12c^2}{4b^2} = \frac{3c^2}{b^2}.$$

By the definition of an exponent (p. 9),

$$a^5 = a \cdot a \cdot a \cdot a \cdot a \text{ and } a^2 = a \cdot a.$$

Then $$a^5 \div a^2 = \frac{\cancel{a} \cdot \cancel{a} \cdot a \cdot a \cdot a}{\cancel{a} \cdot \cancel{a}} = a^3, \text{ or } a^{5-2}.$$

Similarly, $$2^6 \div 2^3 = \frac{\cancel{2} \times \cancel{2} \times \cancel{2} \times 2 \times 2 \times 2}{\cancel{2} \times \cancel{2} \times \cancel{2}} = 2^3, \text{ or } 2^{6-3},$$

and $$ax^3 \div x^2 = \frac{a \cdot x \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x}} = ax, \text{ or } ax^{3-2}.$$

In like manner $6by^5 \div 2y^3 = 3by^2$, or $3b \cdot y^{5-3}$.

These examples illustrate the

Principle. *The exponent of any letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.*

The foregoing principle expressed in general terms is

$$n^a \div n^b = n^{a-b}.$$

What this equation means when $b = a$ and when b is greater than a will be explained later.

From what precedes we see that $ax^2 \div x^2 = \frac{a \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x}} = a.$

Hence a letter which has the same exponent in divisor and dividend should not appear in the quotient.

The law of signs in division may be indicated as follows:

$$+ab \div (+a) = +b.$$
$$+ab \div (-a) = -b.$$
$$-ab \div (+a) = -b.$$
$$-ab \div (-a) = +b.$$

Now $-12\,a \div 6\,b = -\frac{2\,a}{b}$. Here the quotient is a fraction, and the minus sign indicates that the fraction is negative.

Similarly, $$9\,x \div (-3\,y) = -\frac{3\,x}{y},$$

and $$-24\,a^2y \div (-6\,z^3) = +\frac{4\,a^2y}{z^3}.$$

For the division of monomials we have the

Rule. *Divide the numerical coefficient of the dividend by the numerical coefficient of the divisor, keeping in mind the rule of signs for division.*

Write after this quotient all the letters of the dividend except those having the same exponent in divisor and dividend, giving to each letter an exponent equal to its exponent in the dividend minus its exponent in the divisor.

If there are any letters in the divisor unlike those in the dividend, write them under the preceding result as a denominator.

ORAL EXERCISES

Perform the indicated division:

1. $-20 \div 2$.
2. $12 \div (-3)$.
3. $-36 \div (-4)$.
4. $8a^3 \div a^2$.
5. $a^5 \div a^3$.
6. $-a^{14} \div a^7$.
7. $x^6 \div (-x)$.
8. $-x^9 \div (-x^3)$.
9. $ax^3 \div (-ax)$.
10. $-a^2x^5 \div a^2x$.
11. $-a^2xc \div cx$.
12. $a^2x^3c^4 \div a^2x$.
13. $-8a^6 \div 2a^2$.
14. $9x^2 \div (-3x)$.
15. $-18x^7 \div (-6x^4)$.
16. $-35ax^3 \div 5ax^2$.
17. $12ax^3 \div (-3bx^3)$.
18. $14a^3b \div (-2ab)$.
19. $-28ay^4 \div (-7cy^3)$.
20. $-28a^3x^6 \div (-4a^3x^2)$.
21. $70x^6y^9 \div (-14x^2y^8)$.
22. $16ac^4x \div 2axc^2$.
23. $48ax^5 \div (-16bx^4)$.
24. $-36x^6y^6 \div (-6x^2y^3)$.
25. $63c^3d^5 \div (-9cd^3)$.
26. $64a^5b^7 \div (-16ab^7)$.
27. $-28a^6b^{12} \div (-7a^4b^5)$.
28. $-36a^4xb^8 \div 9a^3xb^7$.
29. $\dfrac{15x^2y^4}{75xy^4}$.
30. $\dfrac{42x^{24}y^{56}}{-6x^8y^7}$.
31. $\dfrac{-17a^9b^8}{34a^7b^8}$.
32. $\dfrac{39x^{18}y^{26}}{-13x^{18}y^{13}}$.
33. $\dfrac{-11a^2b^{11}c^{18}}{66ac^{12}}$.
34. $\dfrac{49a^7x^{14}n^{21}}{7a^2x^7n^7}$.

48. Division of a polynomial by a monomial. In multiplying a polynomial by a monomial (p. 73) we multiply each term of the polynomial by the monomial. In division of a polynomial by a monomial we reverse the steps of multiplication and divide each term of the polynomial by the monomial.

Thus $$(ax + bx) \div x = \frac{ax}{x} + \frac{bx}{x} = a + b.$$

Again, $$\frac{6x^4 - 12x^3 + 18x^2}{-3x^2} = -2x^2 + 4x - 6.$$

Therefore, for the division of a polynomial by a monomial we have the

Rule. *Divide each term of the polynomial by the monomial and write the partial quotients in succession.*

ORAL EXERCISES

Perform the indicated division:

1. $\dfrac{a^8 - a^4}{a^2}$.

2. $\dfrac{a^2c - a^8}{a^2}$.

3. $\dfrac{6x^2 - 4x}{2x}$.

4. $\dfrac{9a - 18a^3}{-3a}$.

5. $\dfrac{8x - 12x^3}{-4x}$.

6. $\dfrac{6ax - 10a^2x^2}{2ax}$.

7. $\dfrac{9ax^3 - 12a^3x^5}{-3ax^2}$.

8. $\dfrac{ac + ab}{a}$.

9. $\dfrac{25x^2y + 30xy^5}{-5xy}$.

10. $\dfrac{16bx^4 - 36b^2x^2}{4bx^2}$.

11. $\dfrac{14x^3y^4 - 28x^5y^6}{7x^2y^3}$.

12. $\dfrac{4x^4y - 8x^6y^2 + 12x^8y^4}{4x^4y}$.

13. $\dfrac{a^3cd^2 - a^2c^3 + a^2cd^2}{a^2c}$.

14. $\dfrac{ax^4 - bx^3 + cx^2}{-x^2}$.

15. $\dfrac{15a^2b^2 + 9a^4b^3 - 30a^6b^4}{-3a^2b^2}$.

16. $\dfrac{16a^4b^5 - 24a^5b^6 - 48a^6b^7}{-8a^2b^2}$

17. $\dfrac{85xyz - 51x^2yz^2 + 102x^3yz^3 - 170x^5y^5z}{+17xyz}$.

18. $\dfrac{4(x+3) + a(x+3)}{x+3}$.

19. $\dfrac{2(x+1) + 3x(x+1)}{x+1}$.

20. $\dfrac{(a+x) - 2(a+x)^3}{a+x}$.

21. $\dfrac{(a+b)^4 - 3(a+b)^3}{(a+b)^2}$.

22. $\dfrac{3x(3x+4) - 4y(3x+4)}{3x+4}$.

23. $\dfrac{(a-b)c - (a-b)x}{a-b}$.

24. $\dfrac{21(x-y)^7 - 35(x-y)^5}{-7(x-y)^5}$.

25. $\dfrac{-5(ac^2-2d)^3 + x(ac^2-2d)}{5(ac^2-2d)}$.

49. Division of one polynomial by another. The process of dividing one polynomial by another is illustrated in the following:

EXAMPLES

1. Divide $x^2 - 5x + 6$ by $x - 3$.

Solution.
$$\begin{array}{r|l}
x^2 - 5x + 6 & x - 3 = \text{Divisor} \\ \cline{2-2}
x^2 - 3x & x - 2 = \text{Quotient} \\ \cline{1-1}
-2x + 6 & \\
-2x + 6 & \\ \cline{1-1}
\end{array}$$

Check.
$$\begin{array}{r}
x - 3 \\
x - 2 \\ \hline
x^2 - 3x \qquad \\
-2x + 6 \\ \hline
x^2 - 5x + 6
\end{array}$$

2. Divide $16x + 12x^3 - 15 - 22x^2$ by $2x - 3$.

Solution.
$$\begin{array}{r|l}
12x^3 - 22x^2 + 16x - 15 & 2x - 3 \qquad\quad = \text{Divisor} \\ \cline{2-2}
12x^3 - 18x^2 \qquad\qquad\quad & 6x^2 - 2x + 5 = \text{Quotient} \\ \cline{1-1}
-4x^2 + 16x \qquad & \\
-4x^2 + 6x \qquad & \\ \cline{1-1}
+10x - 15 & \\
+10x - 15 & \\ \cline{1-1}
\end{array}$$

Check.
$$\begin{array}{r}
6x^2 - 2x + 5 \qquad\quad \\
2x - 3 \qquad\quad \\ \hline
12x^3 - 4x^2 + 10x \qquad \\
-18x^2 + 6x - 15 \\ \hline
12x^3 - 22x^2 + 16x - 15
\end{array}$$

The method of dividing one polynomial by another is expressed in the

Rule. *Arrange the dividend and the divisor according to the descending (or ascending) powers of some common letter, called the letter of arrangement.*

Divide the first term of the dividend by the first term of the divisor and write the result as the first term of the quotient.

Multiply the entire divisor by the first term of the quotient, write the result under the dividend, and subtract, being careful to write the terms of the remainder in the same order as those of the divisor.

To find the second term of the quotient, divide the first term of this remainder by the first term of the divisor, and proceed as before until there is no remainder, or until the remainder is of lower degree in the letter of arrangement than the divisor.

Check for division. *The product of the divisor and the quotient should equal the dividend.*

EXERCISES

Divide the following and check Exercises 1, 2, 3, 12, 17, and 18:

1. $x^2+10x+24$ by $x+4$.
2. $x^2-2x-15$ by $x+3$.
3. x^2+x-6 by $x-2$.
4. $3x^2+5x+2$ by $x+1$.
5. $2a^2-3a-2$ by $2a+1$.
6. $5x^2-22x+8$ by $x-4$.
7. $6x^2+19x-7$ by $3x-1$.
8. $3x^2-ax-2a^2$ by $x-a$.
9. $4x^2-8ax+3a^2$ by $2x-3a$.
10. $x^3-8x^2+6x+12$ by $x-2$.
11. $8x^3-12x^2+6x-1$ by $2x-1$.
12. $x^3-11x-6$ by $x+3$.
13. $x^3-14x-8$ by $x-4$.
14. x^3-2x^2-5x+6 by $x-3$.
15. x^3-5x+2 by x^2+2x-1.
16. $x^3-11x+6$ by x^2+3x-2.
17. $x^4-11x^2+2x+12$ by x^2+2x-4.
18. x^3+8 by $x+2$.
19. $8x^3+1$ by $2x+1$.
20. $27x^3+8a^3$ by $3x+2a$.
21. $ax+3a-bx-3b$ by $x+3$.
22. $3ax-ay-6bx+2by$ by $2b-a$.
23. $28ax+9ny-21ay-12nx$ by $3y-4x$.

If there is no remainder, we have seen that the process of division may be expressed as follows:

$$\frac{\text{Dividend}}{\text{Divisor}}=\text{Quotient.}$$

If there is a remainder, the following relation holds:

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Partial quotient} + \frac{\text{Remainder}}{\text{Divisor}}.$$

This last corresponds to what is done in arithmetic in dividing 17 by 5, which is written $\frac{17}{5} = 3\frac{2}{5}$. This means that $\frac{17}{5} = 3 + \frac{2}{5}$, the plus sign being understood.

General check for division. (*a*) When the division is exact. Multiply the divisor by the quotient. The product should be the dividend.

(*b*) When there is a remainder. Multiply the divisor by the partial quotient and add the remainder to the product obtained. The result should be the dividend.

NOTE. We saw on page 1 that it is customary to represent the product of two letters by placing one after the other with no sign between them. Thus ab means a times b. But addition, not multiplication, is implied by placing the fraction $\frac{2}{5}$ after the number 3. This practice comes down to us from the Arabs, who denoted all additions by placing the number symbols in succession without any sign of operation. The later Greeks also had the same notation.

EXERCISES

Divide the following and check Exercises 1, 7, 11, 14, and 22:

1. $6x^2 - 13x + 6$ by $2x - 3$.

2. $25x^4 + 30x^2 - 7$ by $5x^2 + 7$.

3. $6x^2 + 11x - 35$ by $2x + 7$.

4. $19a + 12a^2 - 21$ by $4a - 3$.

HINT. Rearrange the terms in the dividend.

5. $-8 + x^3 + 4x - 2x^2$ by $x - 2$.

6. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.

7. $x^3 - 15x^2 + 65x + 63$ by $x - 7$.

8. $5x^2 + 5x - 25x^3 - 1$ by $5x^2 - 1$.

9. $2x^3 - 14x^2 + 14x + 12$ by $2x - 4$.

10. $6x^3 - 18x + 12$ by $3x - 3$.

11. $4x^3 + 16x + 12$ by $x^2 + x - 3$.

12. $6x^3 - 2x^2 + 10x + 18$ by $3x^2 - 4x + 9$.

13. $3a^3 + 28a^2 + 29a - 140$ by $3a - 5$.

14. $-37x - 6x^3 - 24 + 23x^2$ by $2x - 3$.

15. $2x^2 + 10xy - 4x - 20y$ by $2x + 10y$.

16. $4x^2 + 3cx + 8ax + 6ac$ by $4x + 3c$.

17. $53a + 8 - 53a^2 + 12a^3$ by $4a^2 - 7a - 1$.

18. $-15a^3 + 56a^2 - 99a + 70$ by $3a^2 - 7a + 10$.

19. $3x^4 + 11x^3 - 3x^2 + 17x - 4$ by $x - 3x^2 - 4$.

20. $6an + 15xy - 10ny - 9ax$ by $3x - 2n$.

21. $23a^2 + a^4 - 55a + 11a^3 - 140$ by $a^2 - 5$.

22. $25x^3 - 10x^2 + 36x + 72$ by $5x - 6$.

23. $4a^3 + 1 + a^4 + 4a + 6a^2$ by $1 + a^2 + 2a$.

24. $40x - 31x^2 + 21 + x^4 - 4x^3$ by $x^2 - 3 - 7x$.

25. $a^4 - 8a^3 + 24a^2 - 32a + 16$ by $a^2 - 4a + 4$.

26. $224x - 42x^2 + 10x^4 - 27x^3 - 419$ by $9 + 2x^2 - 5x$.

27. $ax - 6cy - 2bx + 4by + 3cx - 2ay$ by $2y - x$.

28. $x^4 + a^2x^2 + a^4$ by $x^2 - ax + a^2$.

29. $a^4 + 4b^4 + 3a^2b^2$ by $2b^2 + a^2 - ab$.

30. $x^6 - 4x^3y^3 + 8y^6$ by $x^2 + 2y^2 + 2xy$.

31. $9a^4 + 49b^4 + 29a^2b^2$ by $7b^2 + 3a^2 + 4ab$.

32. $x^3 - y^3$ by $x - y$.

33. $a^3 - 125b^3$ by $a - 5b$.

34. $x^4 - 16$ by $x + 2$.

35. $x^6 - 1$ by $x - 1$.

36. $a^6 + 343b^3$ by $a^2 + 7b$.

37. $8a^3 - 64x^9$ by $2a - 4x^3$.

38. $x^5 + y^5$ by $x + y$.

39. $x^5 + y^5$ by $x - y$.

40. $x^5 - y^5$ by $x + y$.

41. $x^5 - y^5$ by $x - y$.

42. $27x^6 + 8n^{12}$ by $3x^2 + 2n^4$.

43. $y^5 - 5y^2 - 3000$ by $y - 5$.

CHAPTER XI

EQUATIONS AND PROBLEMS

50. Equations involving literal coefficients. The most general form of an equation in one unknown is that in which the unknown occurs with literal coefficients. The simplest general form is $ax = b$, in which x is the unknown. The solution of such equations involves no new principle. It is merely necessary to perform the usual operations with letters instead of with ordinary numbers. This should cause no difficulty, since in algebra the letters are really nothing but symbols for numbers, and should be used in all operations as freely as if they were integers.

In solving equations whose coefficients involve letters the answers obtained will usually involve these same letters. Only in exceptional cases may one expect to obtain the root of such an equation in purely numerical form.

In the following exercises the unknown is represented by a letter near the end of the alphabet, as x or y, while the letters in the coefficients and known terms are taken from the beginning of the alphabet.

ORAL EXERCISES

Solve for the unknown:

1. $x - a = 0.$	**6.** $x - a = b.$	**11.** $bx = ab.$
2. $x - 2c = 3c.$	**7.** $3x = 6a.$	**12.** $ax = a^2 + a.$
3. $z + 5b = 7b.$	**8.** $5y = 10c.$	**13.** $2y = 2a + 4b.$
4. $x - a = a.$	**9.** $7z = 21b.$	**14.** $3ay = 6ab.$
5. $x + 3a = a.$	**10.** $ax = a.$	**15.** $3x = 6a + 3b.$

16. $2x - 6a = 4b.$

17. $ay - ab = 2a.$

18. $2ay + 4ab = 10ab.$

19. $(a + b)x = a + b.$

20. $(a + b)y = (a + b)^2.$

21. $(a - b)x = (a - b)(a + b).$

22. $(a + b)(a - c)x = (a + b)(a - c).$

23. $(a - b)(a + b)x = 2ab(a + b)(a - b).$

EXERCISES

Solve for the unknown, and check:

1. $3x - a = x + 5a.$

Solution. $3x - a = x + 5a.$

Transposing, $3x - x = 5a + a.$

Combining, $2x = 6a.$

Dividing by 2, $x = 3a.$

Check. $3 \cdot 3a - a = 3a + 5a,$

or $8a = 8a.$

2. $bx + b = 4b.$

3. $x + a = a + b.$

4. $y - b = a - b.$

5. $3ax + 4a = 7a.$

6. $cz + c^2 = 6c^2.$

7. $ax + ab = ac.$

8. $4bx - b^2 = 5b^2.$

9. $a^2c^2 + 3cx = 7a^2c^2.$

10. $5a^2x + 6a^2 = a^2.$

11. $bc - cx = 4bc.$

12. $5(b + x) = 10b.$

13. $6(c - x) + 18c = 0.$

14. $bx - (b + c) = 5b - c.$

15. $x + 2(b - c) = 4c + 2b.$

16. $3ax - ab = 2ax - ac.$

17. $4cy - 3ac = 5ac + 2cy.$

18. $3cy + 2bc = 6bc + 2cy - 3ac.$

19. $4by - 7a^2b = 6ab^2 + 3by.$

20. $ax - 4a = a^2 + 4 - 2x.$

Solution. $ax - 4a = a^2 + 4 - 2x.$

Transposing, $ax + 2x = a^2 + 4a + 4.$

Writing the coefficients of x as a binomial,

$$(a + 2)x = a^2 + 4a + 4.$$

Dividing both members by the coefficient of x,

$$x = \frac{a^2 + 4a + 4}{a + 2}.$$

Performing the division, $\quad x = a + 2.$

Check. Substituting $a + 2$ for x in the original equation,

$$a(a+2) - 4a = a^2 + 4 - 2(a+2).$$

Expanding, $\quad a^2 + 2a - 4a = a^2 + 4 - 2a - 4,$

or $\quad a^2 - 2a = a^2 - 2a.$

21. $ax + bx = ac + bc.$

22. $5ax + 4cx = 5ab + 4cb.$

23. $a^2x + 1 - a^4 - x = 0.$

24. $ax + 2ab = 2a^2 + bx.$

25. $ax - a^3 - 4 = 3a - x.$

26. $ax - ac + bc = 2ac - 5bc + 2bx.$

27. $4b^2c^2 + (a + bx)c = (a - bx)c.$

Hint. Perform the indicated multiplications first.

28. $(x + a)(x + b) = x^2 + 2a^2 + 3ab.$

29. $15(x - a) - 6(x + a) = 3(5a - 3x).$

51. Uniform motion. If a man walks for 8 hours at the rate of 3 miles per hour, he will walk in all 8×3, or 24, miles. If a train runs for 12 hours at the rate of 45 miles per hour, the total distance traversed is 12×45, or 540, miles. These examples illustrate **uniform motion.** In all problems of uniform motion, the elements involved are

(*a*) Time, measured in seconds, minutes, hours, etc.

(*b*) Rate of motion (velocity), or the distance traveled in a unit of time (one second, one hour, one day, as the case may be).

(*c*) Distance (total), measured in feet, inches, miles, meters, etc.

If a body moves uniformly for a time t, at a rate r, and covers in all a distance d, then the numbers represented by d, r, and t are connected by the equation

$$\boldsymbol{d = r \times t.} \tag{1}$$

This relation or formula gives an insight into the power of the algebraic method. By means of arithmetical numbers alone we can express the relation which holds between the time, the rate, and the distance only for a particular case. By means of the literal equation (1) we express the relation which is true not merely for one case, but for countless cases. In fact, it holds whenever we are dealing with uniform motion, whatever numerical values d, r, and t may have.

The use of letters enables us in this way to express a law in general terms, and hence to include in one short expression a more perfect idea of the relation implied by uniform motion than could be given by many equations involving only arithmetic numbers.

ORAL EXERCISES

1. Sound travels in air about 1100 feet a second. A soldier observes the shower of earth thrown up by an exploding shell, and 10 seconds later hears the sound of the explosion. How far was he from the place where the shell struck?

2. A man observing a woodman fell a tree hears the sound of the ax three fifths of a second after the blow. How far apart are the two men?

3. How far does an automobile travel if it runs

(a) 18 miles per hour for 5 hours?
(b) 16 miles per hour for h hours?
(c) m miles per hour for h hours?
(d) 20 miles per hour for $t + 4$ hours?
(e) $2x - 4$ miles per hour for t hours?

4. What is the rate of an automobile if it runs uniformly

(a) 200 miles in 8 hours?
(b) m miles in 8 hours?
(c) m miles in h hours?
(d) 200 miles in $x + 3$ hours?

5. An automobile travels a distance d at a rate of 20 miles an hour. Another car runs 50 miles farther at 25 miles an hour. Represent the number of hours required by each car for its trip. If the first car required the same time as the other, what equation involving d can be formed?

6. In a naval engagement the opposing battle cruisers were fighting at a distance of ten miles. If the average velocity of the shells was 2640 feet per second, how many seconds did they remain in the air?

7. An automobile runs 20 miles per hour for t hours, and a second one runs 14 miles per hour for $t+2$ hours. Represent the distance each travels. Form an equation which states that the distances traveled by each are the same.

8. James travels t miles per hour for 8 hours, and John travels $t-3$ hours at the rate of 12 miles per hour. Represent the distance each travels. What equation in t may be formed if both travel the same distance?

9. James travels from A to B in t hours at 12 miles an hour. John leaves A just 2 hours after James, and traveling 16 miles per hour reaches B at the same time as James. Represent the distance James travels, the distance John travels, and state the equation in t which may be formed.

10. Two automobiles leave two towns 225 miles apart at the same time and travel toward each other. One travels m miles per hour, while the other travels 5 miles per hour less. They meet in 5 hours. Represent the time, the rate, and the distance for each. What will represent the sum of the distances traveled? What equation in m may be formed?

11. Two cars run 200 miles, one in t hours, the other in 1 hour less time. Represent the rate of each car in miles per hour. If the rate of one is 4 miles per hour more than the other, what equation in t can be formed? If the rate of one is twice the rate of the other?

EXAMPLES IN UNIFORM MOTION

1. A pedestrian traveling 4 miles per hour is overtaken 14 hours after leaving a certain point by a horseman who left the same starting point 8 hours after the pedestrian. Find the rate of the horseman.

Solution. This is a problem in uniform motion, involving the distance, the rate, and the time of a pedestrian and of a horseman respectively. By a careful reading of the problem one discovers that the time for each was a different number of hours, that each went at a different rate, but that each traveled the same distance. Hence the equation will be formed by expressing d in terms of r and t for both the pedestrian and the horseman and then equating the two expressions for d.

By the conditions:

	t, or time in hours	r, or rate in miles per hour	Distance in miles, $d = r \times t$
Pedestrian	14	4	$56 = 4 \times 14$
Horseman	6	x	$6 \cdot x$

Hence $$6x = 56,$$

and $$x = 9\tfrac{1}{3}.$$

Check. $$4 \cdot 14 = 56;\ 9\tfrac{1}{3} \cdot 6 = 56.$$

Biographical Note. *Sir Isaac Newton.* Sir Isaac Newton (1642–1727) was probably the keenest mathematical thinker who ever lived. He was the son of a farmer of slender means, and as a boy was rather lazy. It is said, however, that his complete victory over a larger boy in a fight at school led him to feel that perhaps he could be equally successful in his studies if he really tried. His ambition and interest being once roused, he never ceased to apply himself during the rest of his long life.

His most important scientific achievement was the discovery and verification of the laws of motion. In his great work called the "Principia" he showed by mathematical reasoning that all bodies, great and small,—the planet revolving around the sun, as well as the apple falling from the tree,—follow the same laws. His greatest discovery in pure mathematics was that of a method called the calculus, which is the basis of most of the advances in mathematics and in theoretical physics made since his time.

SIR ISAAC NEWTON

But important as was Newton's mathematical work, his most significant contribution to mankind was an idea, — the idea that the world in which we live is not independent of the rest of the universe, but that every smallest particle of matter is connected with the most remote planet and star; that we cannot think of the earth as the center of all things, but that we merely occupy our place in a system governed by universal law.

2. Two men, A and B, start from the same place at the same time and travel in opposite directions. B goes twice as fast as A. In 9 hours they are 54 miles apart. Find the rate of each.

One can conveniently represent the conditions of this problem in the form of a table, as follows:

	t, or time in hours	r, or rate in miles per hour	Distance in miles, $d = r \times t$
A	9	r	$9\,r$
B	9	$2\,r$	$18\,r$

Since the men are 54 miles apart at the end of the given time, the sum of the distances traveled by A and B is 54.

Hence $$9\,r + 18\,r = 54,$$

or $$r = 2, \text{ and } 2\,r = 4.$$

Check. $4 = 2 \cdot 2;\ 9 \times 2 + 9 \times 4 = 54.$

It should be particularly noted in choosing letters for the unknowns that it is not enough to say, for instance, let x equal the distance, or let t equal the time. This means nothing unless the unit of distance and the unit of time are also stated. The unknown distance is either a number of feet, or miles, or some other unit of length, and the unknown time is a number of seconds, or hours, or some other specific unit of time. A similar remark is pertinent each time a letter is taken to represent the measure of any quantity.

The student should make a table for each of the following problems, similar to those given in the examples.

PROBLEMS

In Problems 1–7, A and B start from the same place at the same time and travel in opposite directions.

1. A goes 6 miles per hour and B goes 9 miles per hour In how many hours will they be 90 miles apart?

2. A travels three times as fast as B. In 6 hours they are 120 miles apart. Find the rate of each.

3. A travels 4 miles more per hour than B. After 8 hours they are 144 miles apart. Find the rate of each.

4. A goes 3 miles less per hour than B. After 9 hours the distance between them is 189 miles. Find the rate of each.

5. B goes 4 miles less per hour than A and travels two thirds as fast as A. Find the rate of each. After how many hours will the distance between them be 180 miles?

6. A travels 2 hours and stops. B travels 5 hours at a rate double A's rate. Then they are 144 miles apart. Find their rates and the distance each has traveled.

7. Both A and B travel the same distance, B in 9 hours, A in 6. B's rate is 4 miles per hour less than A's. Find the rate of each, and the distance each traveled.

In Problems 8–13, A and B start at the same time from two points 192 miles apart and travel toward each other until they meet. Find the rate of each:

8. If they travel at the same rate and meet in 8 hours.

9. If A travels 2 miles less per hour than B and they meet in 12 hours.

10. If B travels three times as fast as A and they meet in 12 hours.

11. If they meet in 9 hours and B travels 42 miles more than A.

12. If they meet in 6 hours and B goes 4 miles more per hour than A.

13. If they meet in 12 hours and A travels 6 miles more per hour than B.

In Problems 14–16, A and B start at the same time from two points 144 miles apart and travel toward each other until they meet. Find the number of hours from the start until the time of meeting:

14. If B goes 4 miles more per hour than A and travels twice as far as A.

15. If A travels 6 miles per hour and B travels 9 miles per hour, but B is delayed 4 hours on the way.

16. If A is delayed 3 hours and B is delayed 5 hours, and their rates are 16 miles and 8 miles per hour respectively.

17. The distance from Kansas City to St. Louis is 285 miles. A passenger train running 45 miles per hour leaves Kansas City for St. Louis at the same time a freight train running 12 miles per hour leaves St. Louis for Kansas City. In how many hours will they meet?

18. A starts from a certain place and travels 4 miles per hour. Six hours later B starts from the same place and travels in the same direction at the rate of 6 miles per hour. How many hours does B travel before overtaking A?

19. Two bicyclists 108 miles apart start at the same time and travel toward each other. One travels 10 miles per hour, the other 12 miles per hour. The latter is delayed 2 hours on the way. In how many hours will they meet, and how far has each traveled?

20. A passenger train starts 2 hours later than a freight train, from the same station but in an opposite direction. The rate of the passenger train is 42 miles per hour and the rate

of the freight train is 24 miles per hour. In how many hours after the passenger train starts will the two trains be 246 miles apart?

21. A messenger going at the rate of 8 miles per hour has journeyed 2 hours when it is found necessary to change the message. At what rate must a second messenger then travel to overtake the first in 8 hours?

22. A man having 4 hours at his disposal wished to ride as far out of town as possible on a trolley car whose rate is 10 miles per hour, and to return on foot at the rate of 3 miles per hour. On the way back he can take a short cut and save one mile. How long a time may he ride on the car?

The velocity of a bullet continually decreases from the instant it leaves the gun. This is due to the resistance of the air. In the following problems consider the velocity of sound to be 1100 feet per second:

23. Two and one-half seconds after a marksman fires his rifle he hears the bullet strike the target, which is 550 yards distant. Find the average velocity of the bullet.

24. One and three-quarters seconds after a marksman fires his revolver he hears the bullet strike the target 50 rods distant. Find the average velocity of the bullet.

CHAPTER XII

IMPORTANT SPECIAL PRODUCTS

52. The square of a binomial. The multiplication

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+\ \ ab \\ \quad +\ \ ab+b^2 \\ \hline a^2+2\,ab+b^2 \end{array}$$

gives the formula

$$(a+b)^2 = a^2 + 2\,ab + b^2.$$

This may be expressed in words as follows:

I. The square of the sum of two terms is the square of the first term plus twice the product of the two terms plus the square of the second term.

Similarly, $$(a-b)^2 = a^2 - 2\,ab + b^2,$$

which may be expressed in words as follows:

II. The square of the difference of two terms is the square of the first term minus twice the product of the two terms plus the square of the second term.

Study the application of I and II in the following:

EXAMPLES

1. $(x+1)^2 = x^2 + 2\,x + 1.$
2. $(y-2)^2 = y^2 - 4\,y + 4.$
3. $(3+a)^2 = 9 + 6\,a + a^2.$
4. $(5-m)^2 = 25 - 10\,m + m^2.$
5. $(3\,a+b)^2 = 9\,a^2 + 6\,ab + b^2.$

ORAL EXERCISES

Expand the following by I or II, page 105:

1. $(a+1)^2$.
2. $(b+2)^2$.
3. $(x-5)^2$.
4. $(y+10)^2$.
5. $(z-12)^2$.
6. $(2+x)^2$.
7. $(6+x)^2$.
8. $(9-d)^2$.
9. $(11-y)^2$.
10. $(x+a)^2$.
11. $(y+b)^2$.
12. $(m-n)^2$.
13. $(a-c)^2$.
14. $(a+2b)^2$.
15. $(3a-x)^2$.
16. $(5x-y)^2$.
17. $(7l+m)^2$.
18. $(12m-n)^2$

State two equal binomials whose product is:

19. m^2+4m+4.
20. $x^2+10x+25$.
21. a^2-6a+9.
22. $d^2-12d+36$.
23. $h^2+14h+49$.
24. $64-16m+m^2$.

Find the value of the following:

25. $(5+2)^2$.

Solution. $(5+2)^2=5^2+2\cdot5\cdot2+2^2=25+20+4=49$.

26. $(10+1)^2$.
27. $(9+2)^2$.
28. $(10-3)^2$.

Solution. $(10-3)^2=10^2-2\cdot10\cdot3+3^2=100-60+9=49$.

29. $(12-2)^2$.
30. $(14-4)^2$.
31. $(21)^2$.

HINT. $(21)^2=(20+1)^2$ etc.

32. $(32)^2$.
33. $(53)^2$.
34. $(103)^2$.
35. $(109)^2$.
36. $(201)^2$.
37. $(504)^2$.
38. $(29)^2$.

HINT. $(29)^2=(30-1)^2$ etc.

39. $(48)^2$.
40. $(98)^2$.
41. $(998)^2$.

EXERCISES

Expand the following:

1. $(2c-3d)^2$.
2. $(5x-2y)^2$.
3. $(3a-2b^2)^2$.
4. $(4x+5y^3)^2$.
5. $(7x^2+2y)^2$.
6. $(5x^2-3y)^2$.
7. $(9a^2+2b^2)^2$.
8. $(11a^2b+2)^2$.
9. $(12ab^2-3c)^2$.
10. $(5a^2b-2a^2)^2$.
11. $(-2a^2c^5+3cd^2)^2$.
12. $(-4xy^2-3xz^2)^2$.

Make any changes which are necessary in the following so that each numerator will be the square of its denominator:

13. $\dfrac{x^2+2xy+y^2}{x+y}$.

14. $\dfrac{m^2-mn+n^2}{m-n}$.

15. $\dfrac{a^2+10a+25}{a+5}$.

16. $\dfrac{x^2+12x+36}{x-6}$.

17. $\dfrac{4+4x+x^2}{2+x}$.

18. $\dfrac{25-10x-x^2}{5-x}$.

19. $\dfrac{m^2+16m-64}{8+m}$.

20. $\dfrac{a^4+10a^2-25}{a^2+5}$.

21. $\dfrac{9x^2-6x+1}{3x+1}$.

22. $\dfrac{4a^2+12ab+9b^2}{2a-3b}$.

23. $\dfrac{4x^4-20x^2y+25y^2}{2x^2-5y}$.

24. $\dfrac{4x^4-20x^2y+25y^2}{5y-2x^2}$.

53. The product of the sum and the difference of two terms. The multiplication

$$\begin{array}{lll} a & +b & \\ a & -b & \\ \hline a^2 & +ab & \\ & -ab & -b^2 \\ \hline a^2 & & -b^2 \end{array}$$

gives the formula $(a+b)(a-b)=a^2-b^2$.

This may be expressed in words as follows:

III. The product of the sum and the difference of two terms equals the difference of their squares taken in the same order as the difference of the terms.

The pupil should study the application of III in the following:

EXAMPLES

1. $(x+1)(x-1)=x^2-1$.
2. $(5-x)(5+x)=25-x^2$.
3. $(a+x)(a-x)=a^2-x^2$.
4. $(2c-y)(2c+y)=4c^2-y^2$.
5. $(c^2-3)(c^2+3)=c^4-9$.

ORAL EXERCISES

Perform the following indicated operations:

1. $(a+2)(a-2)$.
2. $(m+7)(m-7)$.
3. $(h+9)(h-9)$.
4. $(6+t)(6-t)$.
5. $(1-x)(1+x)$.
6. $(c+d)(c-d)$.
7. $(x+3y)(x-3y)$.
8. $(5x+2y)(5x-2y)$.
9. $(1+4x)(1-4x)$.
10. $(9a-1)(9a+1)$.
11. $(2+3r)(2-3r)$.
12. $(x^2-4)(x^2+4)$.
13. $(7x^2+2y)(7x^2-2y)$.
14. $(4x^2+3x^3y)(4x^2-3x^3y)$.
15. $(2a+3bc^2)(2a-3bc^2)$.
16. $(a+3a^2b)(a-3a^2b)$.
17. $(7a+2x)(2x-7a)$.
18. $(5x+3y)(3y-5x)$.
19. $(5m^2-4n)(5m^2+4n)$.
20. $(2ab+3c)(2ab-3c)$.
21. $(9cd^2-2)(9cd^2+2)$.
22. $(10mn^2-3p)(10mn^2+3p)$.
23. $(4a^2-9b^2)\div(2a+3b)$.
24. $(9x^2-25y^2)\div(3x-5y)$.
25. $(4a^2b^2-c^2)\div(2ab-c)$.
26. $(49-x^4)\div(7+x^2)$.
27. $(25x^4-4)\div(5x^2+2)$.
28. $(36a^4-49)\div(7+6a^2)$.

State two binomials whose product is:

29. m^2-9.
30. c^2-16.
31. $49-m^2$.
32. m^2-49.
33. $81-p^2$.
34. $100-h^2$.
35. a^2-4x^2.
36. x^2-9y^2.
37. $25x^2-4y^2$.
38. $9a^2-16b^2$.
39. $36c^2-1$.
40. $1-49x^2$.

Find the value of:

41. $(12-2)(12+2)$.
42. $(40+3)(40-3)$.
43. $(30+1)(30-1)$.
44. $(50-2)(50+2)$.
45. 22×18.

HINT. $22\times18=(20+2)(20-2)$ etc.

46. 13×7.
47. 23×17.
48. 43×37.
49. 64×56.
50. 71×69.
51. 83×77.
52. 96×104.
53. 91×109.

54. The product of two binomials having a common term. The multiplication

$$\begin{array}{l} x + a \\ \underline{x + b} \\ x^2 + ax \\ \underline{\quad + bx + ab} \\ x^2 + (a + b)x + ab \end{array}$$

gives the formula

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

This may be expressed in words as follows:

IV. The product of two binomials having a common term equals the square of the common term, plus the algebraic sum of the unlike terms multiplied by the common term, plus the algebraic product of the unlike terms.

The pupil should study the application of IV in the following:

EXAMPLES

1. $(x + 1)(x + 2) = x^2 + (1 + 2)x + 2 = x^2 + 3x + 2.$
2. $(c - 2)(c - 3) = c^2 + (-2 - 3)c + 6 = c^2 - 5c + 6.$
3. $(n - 5)(n + 2) = n^2 + (-5 + 2)n - 10 = n^2 - 3n - 10.$

ORAL EXERCISES

Multiply the following mentally:

1. $(x + 2)(x + 1).$
2. $(c + 3)(c + 2).$
3. $(h + 4)(h + 5).$
4. $(a + 5)(a + 3).$
5. $(a + 7)(a + 2).$
6. $(c + 5)(c + 4).$
7. $(x - 2)(x - 3).$
8. $(c - 3)(c - 4).$
9. $(x - 5)(x - 2).$
10. $(x - 7)(x - 8).$
11. $(c - 8)(c + 9).$
12. $(c + 10)(c - 3).$
13. $(c - 7)(c + 5).$
14. $(x + a)(x + c).$
15. $(x + k)(x + l).$
16. $(x + a)(x - c).$

EXERCISES

Write the quotients for the following by inspection and check by multiplication:

1. $\dfrac{x^2+5x+6}{x+3}$.
2. $\dfrac{a^2+7a+10}{a+2}$.
3. $\dfrac{a^2+11a+10}{a+10}$.
4. $\dfrac{b^2+7b+6}{b+1}$.
5. $\dfrac{c^2-6c+8}{c-2}$.
6. $\dfrac{c^2-9c+14}{c-7}$.
7. $\dfrac{x^2-3x+2}{x-2}$.
8. $\dfrac{x^2-4x-12}{x+2}$.
9. $\dfrac{a^2-3a-28}{a-7}$.
10. $\dfrac{a^2-3ab+2b^2}{a-b}$.
11. $\dfrac{x^2-xy-6y^2}{x-3y}$.
12. $\dfrac{m^2-4mn-21n^2}{m+3n}$.
13. $\dfrac{r^2-7rs-18s^2}{r-9s}$.
14. $\dfrac{r^2-rs-90s^2}{r-10s}$.

55. The product of two binomials of the form $(ax+b)(cx+d)$. The multiplications

$$\begin{array}{r} 2x+5 \\ 3x+7 \\ \hline 6x^2+15x \qquad\quad \\ +14x+35 \\ \hline 6x^2+29x+35 \end{array} \quad \text{and} \quad \begin{array}{r} ax+b \\ cx+d \\ \hline acx^2+bcx \qquad\quad \\ +adx+bd \\ \hline acx^2+(bc+ad)x+bd \end{array}$$

show that in the product of any two binomials of the form $(ax+b)(cx+d)$

(*a*) the first term is the product of the first terms of the binomials,

(*b*) the second term is the sum of the cross products, and

(*c*) the third term is the algebraic product of the second terms of the binomials.

EXAMPLES

1. $(3x+2)(4x+5)=12x^2+8x+15x+10$
$\quad\quad\quad\quad\quad\quad\quad\;\;=12x^2+23x+10.$

2. $(5a+1)(3a-2)=15a^2+3a-10a-2$
$\quad\quad\quad\quad\quad\quad\quad\;\;=15a^2-7a-2.$

3. $(4b-3)(b+2)=4b^2+5b-6.$

EXERCISES

Write the products in the following:

1. $(2a+1)(a+3)$.
2. $(3a+2)(a+5)$.
3. $(x+4)(2x+3)$.
4. $(2x+3)(x+7)$.
5. $(2c+3)(c+8)$.
6. $(4c+3)(4c+3)$.
7. $(5m+2)(5m+2)$.
8. $(4m-1)(2m+3)$.
9. $(5n-3)(3n-5)$.
10. $(2n+7)(3n-2)$.
11. $(7h+3)(4h-5)$.
12. $(5h-4)(5h-4)$.
13. $(5k-6)(5k-6)$.
14. $(3+m)(4+2m)$.
15. $(7-n)(7-3n)$.
16. $(2a-b)(3a+2b)$
17. $(5a+9c)(2a-9c)$.
18. $(2x^2-7y)(3x^2-2y)$.
19. $(c^2-5d)(5c^2+2d)$.
20. $(4c+9d^2)(2c-5d^2)$.

ORAL REVIEW EXERCISES

Perform the following indicated operations:

1. $(m+2r)^2$.
2. $(h-3k)^2$.
3. $(2a+13)^2$.
4. $(6s-5r)^2$.
5. $(a^2+3x)^2$.
6. $(2m-3n^2)^2$.
7. $(a+3c)(a-3c)$.
8. $(5a+1)(5a-1)$.
9. $(5n^2-p^2)^2$.
10. $(3rs^2+2s)^2$.
11. $(2x+3y)(2x-3y)$.
12. $(4h^2k-5h^3)^2$.

13. $(5x^2 - z)(5x^2 + z)$.

14. $(-5a^2c^3 - 2c)^2$.

15. $(31)^2$.

16. $(49)^2$.

17. $(a + 3)(a + 2)$.

18. $(a - 9)(a - 4)$.

19. $(7m^2n + 3)(7m^2n - 3)$.

20. $(4xy^2 + 5z^2)(4xy^2 - 5z^2)$.

21. $(x + y)(x^2 + y^2)(x - y)$.

HINT. Multiply the product of the first and third binomials by the second binomial.

22. $(a - 2)(a^2 + 4)(a + 2)$.

23. $(m + n)(m^2 + n^2)(m - n)(m^4 + n^4)$.

24. $(x^2 + 1)(x^2 - 1)(x^4 + 1)(x^8 + 1)$.

25. $[(a + b) + c][(a + b) - c]$.

26. $[(x + y) + 2][(x + y) - 2]$.

27. $(x + 2a)(x + a)$.

28. $(2c + 1)(c + 2)$.

29. $(3d^2 + 4)(2d^2 + 5)$.

30. $(x - y)(x + 3y)$.

31. $(c - 3d)(c - d)$.

32. $(h^2 + 2k)(h^2 - 3k)$.

33. $(3a^2 + 4)(3a^2 + 4)$.

34. $(a^2 + 3b)(2a^2 - b)$.

35. $(5x^2 - y)(5x^2 - y)$.

36. $(4c^2 + d^2)(4c^2 - d^2)$.

37. $(4x^2 - 2a^3)(4x^2 - 3a^3)$.

38. $(5x - y^3z^2)(2x + 3y^3z^2)$.

39. $(a^2 + 2am + m^2) \div (a + m)$.

40. $(x^2 - 4y^2) \div (x - 2y)$.

41. $(a^2 - 3a + 2) \div (a - 2)$.

42. $(49 - 14x + x^2) \div (7 - x)$.

43. $(c^2 - 5c + 6) \div (c - 3)$.

44. $(9c^2 - 4) \div (3c + 2)$.

45. $(d^2 - 16d - 17) \div (d + 1)$.

What is the quickest way to multiply:

46. $(x + y)(x + y)(x - y)(x - y)$?

47. $(x^2 + a^2)(x + a)(x^4 + a^4)(x - a)$?

48. $(4x^2 + a^2)(2x + a)(16x^4 + a^4)(2x - a)$?

CHAPTER XIII

FACTORING

56. Definitions. **Factoring** is the process of finding two or more expressions whose product is equal to a given expression.

Many simple exercises in factoring were solved in the preceding chapter in connection with the rules of multiplication there given. In fact the process of factoring is the reverse of multiplication.

The subject of factoring is extensive. In this chapter we shall consider only the more common forms of factorable expressions, using only such factors as have integers as coefficients.

If a polynomial cannot be expressed as the product of expressions other than itself and 1, it is said to be **prime.**

57. Square root of monomials. In factoring it is often necessary to find the square root, the cube root, and other roots of monomials.

The **square root** of a monomial is one of the two equal factors whose product is the monomial.

Since $+2 \cdot +2 = 4$ and $-2 \cdot -2 = 4$, the square root of 4 is ± 2, which means both plus 2 and minus 2.

Similarly, the square root of 9 is ± 3 and the square root of a^2 is $\pm a$.

That is, *Every positive number or algebraic expression has two square roots which have the same absolute value but opposite signs.*

It is customary to speak of the positive square root of a number as the **principal square root**, and if no sign precedes the radical, the principal root is understood.

Thus $\sqrt{4} = 2$, not -2; $-\sqrt{4} = -2$, not $+2$.

When both the positive and the negative square roots are considered, the double sign must precede the radical.

Since $x^3 \cdot x^3 = (-x^3)(-x^3) = x^6$, then $\pm\sqrt{x^6} = \pm x^3$.

That is, *The exponent of any letter in the square root of a monomial is one half the exponent of that letter in the monomial.*

Hence for extracting the square root of a monomial where both positive and negative factors are desired we have the

Rule. *Write the square root of the numerical coefficient preceded by the double sign* $\pm$ *and followed by all the letters of the monomial, giving to each letter an exponent equal to one half its exponent in the monomial.*

A rule much like the preceding holds for fourth root, sixth root, and other even roots.

Thus $\pm\sqrt[4]{81\,c^8} = \pm 3\,c^2$, and $\pm\sqrt[6]{a^{18}} = \pm a^3$.

In the chapters on Factoring and Fractions, where square roots arise, only the *principal* square root will be considered.

According to the definition of square root, the two factors of a term, either of which is its square root, *must be equal.* Consequently they must have the same sign.

58. Cube root of monomials. The **cube root** of a monomial is one of the three equal factors whose product is the monomial.

Since $3 \cdot 3 \cdot 3 = 27$, $\sqrt[3]{27} = 3$.

And as $-3 \cdot -3 \cdot -3 = -27$, $\sqrt[3]{-27} = -3$.

That is, *The cube root of a monomial has the same sign as the monomial.*

Since $x^5 \cdot x^5 \cdot x^5 = x^{15}$, $\sqrt[3]{x^{15}} = x^5$.

Similarly, as $a^5x^2 \cdot a^5x^2 \cdot a^5x^2 = a^{15}x^6$, $\sqrt[3]{a^{15}x^6} = a^5x^2$.

That is, *The exponent of any letter in the cube root of a term is one third of the exponent of that letter in the term.*

Hence for extracting the cube root of a monomial we have the

Rule. *Write the cube root of the numerical coefficient preceded by the sign of the monomial and followed by all the letters of the monomial, giving to each letter an exponent equal to one third of its exponent in the monomial.*

A rule much like the preceding holds for fifth root, seventh root, and other odd roots.

Thus $\sqrt[5]{-32} = -2$, $\sqrt[5]{x^{10}} = x^2$, and $\sqrt[7]{128\,x^{21}} = 2\,x^3$.

ORAL EXERCISES

Find the value of the following:

1. $\sqrt{3^2}$.
2. $\sqrt{7^2}$.
3. $\sqrt{6^4}$.
4. $\sqrt{5^2 \cdot 2^2}$.
5. $\sqrt{9 \cdot 5^2}$.
6. $\sqrt{4\,x^2}$.
7. $\sqrt{9\,a^4}$.
8. $\sqrt{16\,x^2}$.
9. $\sqrt{25\,x^8}$.
10. $\sqrt{4\,a^2x^6}$.
11. $\sqrt{36\,a^4c^2}$.
12. $\sqrt{64\,m^6n^{10}}$.
13. $\sqrt{49\,r^2s^{14}}$.
14. $\sqrt{100\,a^{10}b^2}$.
15. $\sqrt{144\,a^8y^{10}}$.
16. $\sqrt{196\,h^6k^{16}}$.
17. $\sqrt{121\,a^{20}b^{10}c^2}$.
18. $\sqrt{225\,c^{12}d^6}$.
19. $\sqrt{289\,a^2b^4c^6}$.
20. $\sqrt{361\,x^2y^{12}z^{20}}$.
21. $\sqrt{225\,m^{16}n^8z^4}$.
22. $\sqrt{2^2 \cdot 3^4 \cdot 5^2}$.
23. $\sqrt{10^4 \cdot 6^2 \cdot x^{10}}$.
24. $\sqrt{5^2 \cdot 2^4 \cdot 9}$.
25. $\sqrt{9 \cdot 25 \cdot 81}$.
26. $\sqrt{m^2(n^2)^2}$.
27. $\sqrt{a^6(b^8)^2}$.
28. $\sqrt{4\,x^2(y^2)^2}$.
29. $\sqrt{2^2 \cdot (7^2)^2}$.
30. $\sqrt{3^2 \cdot (2^2)^4}$.
31. $\sqrt{x^4 \cdot (y^{10})^2}$.
32. $\sqrt{25 \cdot 2^6 \cdot r^4}$.
33. $2\sqrt{x^{10}}$.
34. $5\sqrt{a^6 \cdot 2^{10}}$.
35. $7\sqrt{x^2 \cdot 3^2 \cdot y^6}$.
36. $\sqrt[3]{5^3}$.

37. $\sqrt[3]{(-2)^6}$.
38. $\sqrt[3]{(-8)\cdot 2^3}$.
39. $\sqrt[3]{5^3\cdot 3^6}$.
40. $\sqrt[3]{8\cdot 3^3}$.
41. $\sqrt[3]{-27\cdot 8}$.
42. $\sqrt[3]{27\cdot x^3}$.
43. $\sqrt[3]{-64\cdot a^9}$.
44. $\sqrt[3]{125\,x^{12}}$.
45. $\sqrt[3]{343\cdot a^{33}b^3}$.
46. $\sqrt[3]{(-x)^3\cdot 5^3}$.
47. $\sqrt[3]{(-3)^6\cdot 8^2\cdot c^9}$.
48. $\sqrt[3]{1000\cdot r^6}$.
49. $\sqrt[3]{-729\cdot 9^3\cdot s^{12}}$.
50. $\sqrt[4]{a^4b^8}$.
51. $\sqrt[4]{x^{12}y^{16}}$.
52. $\sqrt[4]{2^4\,x^{20}}$.
53. $\sqrt[4]{3^4 5^8}$.
54. $\sqrt[4]{16}$.
55. $\sqrt[5]{32\,x^{10}}$.
56. $\sqrt[5]{-243\,a^{15}}$.
57. $\sqrt[5]{4\cdot 2^3a^{10}b^{15}}$.

59. Polynomials with a common monomial factor. The type form is

$$ab + ac - ad.$$

Since
$$ab + ac - ad = a(b + c - d),$$

we have, for factoring expressions having a common monomial factor, the following

Rule. *Determine by inspection the monomial factor which is the product of all numerical and literal factors common to all terms of the polynomial.*

Divide the polynomial by this monomial factor.

Write the quotient in a parenthesis preceded by the monomial factor.

EXAMPLE

Factor $10\,ab^2 - 15\,b^3$.

Solution. The common monomial factor of both terms is seen to be $5\,b^2$. Dividing the binomial by $5\,b^2$, the quotient is $2\,a - 3\,b$.

Therefore $10\,ab^2 - 15\,b^3 = 5\,b^2(2\,a - 3\,b)$.

ORAL EXERCISES

Factor the following:

1. $2x + 2$.
2. $3x + 6$.
3. $2a - 4$.
4. $3c + 9$.
5. $5c - 10$.
6. $7x - 14$.
7. $10x + 20$.
8. $5m - 15$.
9. $10 - 2n$.
10. $12 - 6a$.
11. $2b + 2c$.
12. $5r - 10s$.
13. $6x - 4y$.
14. $10x + 2y$.
15. $16a - 4b$.
16. $20c + 18cd$.
17. $25m - 20mn$.
18. $27x - 36xy$.
19. $30r - 12rs$.
20. $35t + 4st$.

EXERCISES

Write the prime factors of the following:

1. $3ax^2 + 18a$.
2. $8x^3 + 8x$.
3. $2x^4 - 6x^2$.
4. $a^3b^2 + a^3b^3$.
5. $3r^2s - 27r^2$.
6. $10x^8 - 4x^7$.
7. $18x^2 + 27 + 9x$.
8. $x^3 - x^2 + x^4$.
9. $2c^5 - 18c + 2c^3$.
10. $4x^2 - 8ax + 20x$.
11. $5a^2 + 10a^4 - 25a^3$.
12. $3y^4 + 6y^3c - 3y^3$.
13. $a^2 + a^3 - a^4 + 2a$.
14. $5r^2 + 10r + 15r^3$
15. $6a^5 - 12a^4 + 6a - 18a^6$.
16. $16m^3 - 32m^2n + 24m^2n^2$.
17. $-a^2b - 2ab^2 + a^2b^2$.
18. Solve for x, $ax = a(m + n)$.
19. Solve for x, $ax = ab + ac$.
20. Solve for y, $ny = nr + ns + nt$.
21. Solve for y, $my = mh - mk - ml$.
22. Solve for z, $az = 2a - ab + a^2$.

60. Polynomials which may be factored by grouping terms and taking out a common binomial factor. The type form is

$$ax + ay + bx + by.$$

Plainly $\quad ax + ay + bx + by = a(x + y) + b(x + y)$.

Dividing both terms of $a(x + y) + b(x + y)$ by $(x + y)$, the quotient is $a + b$.

Therefore $\quad ax + ay + bx + by = (x + y)(a + b)$.

EXAMPLE

Factor $2xy + 3ab + 6ay + bx$.

Solution.

$$2xy + 3ab + 6ay + bx =$$
$$2xy + 6ay + bx + 3ab =$$
$$2y(x + 3a) + b(x + 3a) =$$
$$(x + 3a)(2y + b).$$

The preceding example illustrates the

Rule. *Arrange the terms of the polynomial to be factored in groups of two or more terms each, such that in each group a monomial factor may be written outside a parenthesis, which in each case contains the same expression.*

Rewrite, placing these monomial factors outside parentheses.

Then divide by the expression in parenthesis and write the divisor as one factor and the quotient as the other.

Polynomials which may be factored by grouping terms according to the foregoing rule usually contain either four, six, or eight terms.

It is important to note that one can obtain two apparently different sets of factors for a given expression. Thus

$$(m-r)(n-2x)=(r-m)(2x-n),$$

for each pair by actual multiplication gives $mn - nr - 2mx + 2rx$.

An inspection of the expression shows that the binomials of the first pair are the negatives respectively of those in the second pair; hence either pair of factors is correct.

The relation that the process of factoring bears to the processes of multiplication and division of monomials and polynomials should be constantly kept in mind. In multiplication we have two factors given and are required to find their product. In division we have the product and one factor given and are required to find the other factor. In factoring, however, the problem is a little more difficult, for we have only the product given, and our experience is supposed to enable us to determine the factors. For this reason a very careful study of various typical forms of products is necessary.

There is no simple operation the performance of which makes us sure that we have found the *prime* factors of a

given expression. Only insight and experience enable us to find prime factors with certainty.

A partial check, however, that may be applied to all the exercises in factoring, consists in actually multiplying together the factors that have been found. The result should be the original expression.

ORAL EXERCISES

Factor the following:

1. $5(x+y)+c(x+y)$.
2. $a(2b+c)+c(2b+c)$.
3. $m(a+3x)-n(a+3x)$.
4. $2x(x-2y)-y(x-2y)$.
5. $3r(s+7t)-(s+7t)$.
6. $2m(m+n)+3n(n+m)$.
7. $3x(2x-y)-5y(2x-y)$.
8. $m(m-5n)-2n(m-5n)$.
9. $2a(a-b)-3b(a-b)$.
10. $5x(x-3)+3(x-3)$.
11. $4r(r-7)-7(r-7)$.
12. $7s(s-t)-2t(s-t)$.
13. $9m(m-2n)+4n(m-2n)$.
14. $7x(3x-4y)-2y(3x-4y)$.
15. $11ab(a-3b)-5c(a-3b)$.

EXERCISES

Factor the following:

1. $2a(a-x)+3(x-a)$.

Hint. This can be written $2a(a-x)-3(a-x)$.

2. $7m(m-2n)-2(2n-m)$.
3. $3h(5h-k)-k(k-5h)-hk(5h-k)$.
4. $7x(2x-5y)-2y(2x-5y)+(5y-2x)$.
5. $(3a-2x)+9a(-2x+3a)+x(2x-3a)$.
6. $ah+bh+ak+bk$.

7. $mr + nr + ms + ns.$

8. $2\,ac + bc + 2\,ad + bd.$

9. $3\,ac + cx + 9\,ay + 3\,xy.$

10. $6\,ax + 15\,bx + 4\,ay + 10\,by.$

11. $2\,a^3 + a^2x + 2\,ax + x^2.$

12. $6\,h^3 + 4\,h^2k + 3\,hk^2 + 2\,k^3.$

13. $mh - nh + mk - nk.$

14. $2\,ax - 6\,ay + bx - 3\,by.$

15. $ar - cr + as - cs.$

16. $mr - nr + ms - ns.$

17. $2\,hx - 2\,kx + hy - ky.$

18. $3\,x^3 - 3\,x^2y + xy - y^2.$

19. $2\,a^3 - 6\,a^2b^2 + ab^3 - 3\,b^5.$

20. $5\,r^3 - 2\,rs^2 + 10\,r^2s - 4\,s^3.$

21. $20\,x^3 - 5\,x^2y^2 + 4\,xy - y^3.$

22. $6\,ac - 3\,c + 2\,ad - d.$

23. $8\,ax + 15\,by - 12\,ay - 10\,bx.$

Solution. $8\,ax + 15\,by - 12\,ay - 10\,bx =$
$8\,ax - 12\,ay - 10\,bx + 15\,by =$
$4\,a(2\,x - 3\,y) - 5\,b(2\,x - 3\,y) =$
$(2\,x - 3\,y)(4\,a - 5\,b).$

24. $ac - bc - ad + bd.$

25. $ac - ad - bc + bd.$

26. $2\,am - 6\,bm - 3\,an + 9\,bn.$

27. $3\,hx - 15\,kx - hy + 5\,ky.$

28. $x^3 - 3\,xy - 2\,x^2y^3 + 6\,y^4.$

29. $2\,a^3 - 6\,a^2b^2 - ab^2 + 3\,b^4.$

30. $10\,a^4 - 25\,a^3b^3 + 5\,b^4 - 2\,ab.$

31. $14\,x^4 - 35\,ax^2 + 10\,a - 4\,x^2.$

32. $30\,x^5 - 10\,x^2 + 1 - 3\,x^3.$

33. $ab^4 - 2\,b^3 - 5\,ab + 10.$

34. Solve for x, $x(a + b) = r(a + b) + s(a + b)$.

35. Solve for x, $mx + nx = mr + nr + ms + ns$.

HINT. In the preceding equation, and in similar ones, the value of the unknown should be obtained by the use of factoring. Mental division should be employed, not ordinary long division.

36. Solve for y, $ay + by = ac + bc - ab - b^2$.

37. Solve for z, $kz - lz = hk - hl - k^2 + lk$.

38. Solve for y, $cy + ad - ae = dc - ce + ay$.

39. Solve for m, $2\,k + hl - mk = kl + 2\,h - mh$.

61. Trinomials which are perfect squares. Here the type form is

$$a^2 \pm 2ab + b^2.$$

This, by page 105, gives us the two expressions:

$$a^2 + 2ab + b^2 = (a+b)^2,$$
$$a^2 - 2ab + b^2 = (a-b)^2.$$

If an algebraic expression is the product of two equal factors, it is said to be a **perfect square.**

A trinomial, arranged according to the descending powers of one letter, *is a perfect square if the first and third terms are positive and if the absolute value of the middle term is twice the product of the absolute values of the square roots of the other two terms.*

Thus in the type form above, the middle term $2ab = 2 \cdot \sqrt{a^2} \cdot \sqrt{b^2}$.

Similarly, the trinomial $4x^2 - 20xy^2 + 25y^4$ is a perfect square, since the middle term $20xy^2 = 2 \cdot \sqrt{4x^2} \cdot \sqrt{25y^4} = 2 \cdot 2x \cdot 5y^2$.

ORAL EXERCISES

Form perfect trinomial squares by supplying the missing terms in the following:

1. $a^2 + (?) + x^2$.
2. $c^2 + (?) + 25$.
3. $s^2 + (?) + 16t^2$.
4. $h^2 + (?) + 1$.
5. $1 + (?) + 4h^2$.
6. $9x^2 - (?) + 4$.
7. $c^2 + 2cd + (?)$.
8. $a^2 - 2ax + (?)$.
9. $y^2 - 6y + (?)$.
10. $r^2 + 2r + (?)$.
11. $x^2 - 4x + (?)$.
12. $x^2 + 10x + (?)$.
13. $c^2 - 8c + (?)$.
14. $n^2 - 12n + (?)$.
15. $n^2 - 20n + (?)$.
16. $x^2 - 18x + (?)$.
17. $x^2 - 26x + (?)$.
18. $4r^2 - 4r + (?)$.
19. $9x^2 + 6x + (?)$.
20. $9a^2 - 6a + (?)$.

21. $16x^2 - 24x + (?)$.

22. $16x^2 - 40x + (?)$.

23. $25x^2 - 60x + (?)$.

24. $(?) + 10r + 25$.

25. $(?) - 18x + 81$.

26. $(?) - 8x + x^2$.

For obtaining *one* of the two equal factors of a perfect trinomial square we have the

Rule. *Arrange the terms of the trinomial according to the descending powers of some letter in it.*

Extract the square root of the first and third terms and connect the results by the sign of the middle term.

Before applying the foregoing rule one should never forget to observe whether the expression to be factored is a perfect trinomial square or not.

ORAL EXERCISES

Factor the following:

1. $x^2 + 4x + 4$.
2. $b^2 + 6b + 9$.
3. $c^2 - 8c + 16$.
4. $a^2 + 10a + 25$.
5. $d^2 - 10d + 25$.
6. $h^2 - 12h + 36$.
7. $49 - 14x + x^2$.
8. $64 + 16m + m^2$.
9. $81 - 18m + m^2$.
10. $n^2 - 20n + 100$.
11. $k^2 - 22k + 121$.
12. $144 - 24y + y^2$.
13. $c^2 + 2c + 1$.
14. $1 - 2x + x^2$.
15. $4x^2 + 4x + 1$.
16. $9c^2 + 6c + 1$.
17. $9c^2 + 12c + 4$.
18. $16x^2 + 8x + 1$.
19. $16x^2 - 24x + 9$.
20. $25x^2 - 20x + 4$.
21. $9 + 42x + 49x^2$.
22. $16 - 24x + 9x^2$.
23. $25 - 40x + 16x^2$.
24. $49 + 70x + 25x^2$.

EXERCISES

Write the factors for the following:

1. $9r^2 + 6rs + s^2$.
2. $4a^2 + 4ax + x^2$.
3. $9x^2 - 6xy + y^2$.
4. $16x^2 - 8xm + m^2$.
5. $m^2 + 10mn + 25n^2$.
6. $25y^2 - 10yx + x^2$.
7. $81a^2 + 18ab + b^2$.
8. $m^2 - 26mn + 169n^2$.
9. $9h^2 - 60hk + 100k^2$.
10. $121a^2 - 44ab + 4b^2$.
11. $81x^2 + 126xy + 49y^2$.
12. $36x^2 + 25y^2 - 60xy$.
13. $169a^2 + 9b^2 - 78ab$.
14. $49d^2 + 210cd + 225c^2$.
15. $196a^2 - 140ab + 25b^2$.
16. $9a^2b^2 - 12ab + 4$.
17. $16c^2x^2 + 56cx + 49$.
18. $9m^2n^2 - 24mnp + 16p^2$.

It is only in the beginning of factoring that polynomials are classified for the student. In the practical work of handling fractions and solving equations he must determine for himself the type of the polynomial to be factored. It is therefore very important that he fix in mind the various types and the manner of factoring each. Moreover, he should remember that the polynomials which arise in practice often have three or more factors. Miscellaneous review exercises afford excellent practice in recognizing types and in determining *all* the prime factors.

At this point the suggestions given on page 135 will prove helpful, though only the first three of the types there given have as yet been considered.

REVIEW EXERCISES

Separate into prime factors:

1. $x^3 + 6x^2 + 9x$.
2. $2x^2 + 4x + 2$.
3. $x^3 - 10x^2 + 25x$.
4. $a^3 + 2a^2b + ab^2$.
5. $a^3 + 2a^2x + ax^2$.
6. $2c^3 - 20c^2 + 50c$.
7. $100x - 80x^2 + 16x^3$.
8. $98c + 28c^2 + 2c^3$.

9. $80r - 40r^2 + 5r^3$.

10. $245x^5 - 140x^4 + 20x^3$.

11. $45m^2 - 60mn + 20n^2$.

12. $2ax + 4ax^2 + 2ax^3$.

13. $126c^2d^2 + 147c^3d + 27cd^3$.

14. $2ac + 4cx + 2ay + 4xy$.

15. $2ax - 6bx + 2ay - 6by$.

16. $3mr - 6nr + 15ms - 30ns$.

17. $12ab - 6bc - 6ax + 3cx$.

18. Solve for x, $ax + cx = m(a + c) - n(a + c)$.

19. Solve for x, $(a + b)x = a^2 + 2ab + b^2$.

20. Solve for m, $mh - mk = h^2 - 2hk + k^2$.

21. Solve for n, $nr - 2ns = r^2 - 4rs + 4s^2$.

22. Solve for y, $ky - 4kl - l^2 = 4k^2 - ky - ly$.

23. Solve for y, $acy - ad^2 - ac^2 = ady - 2acd$.

24. Solve for z, $2ade - ae^2 = ad^2 - adz + aez$.

62. A binomial the difference of two squares. The type form is

$$a^2 - b^2.$$

By page 107, $\quad a^2 - b^2 = (a + b)(a - b)$.

Hence we have the

Rule. *Regarding each term of the binomial as positive, extract its square root.*

Add the two square roots for one factor, and subtract the second from the first for the other.

EXAMPLES

1. Factor $c^4 - d^4$.

Solution. $c^4 - d^4 = (c^2 + d^2)(c^2 - d^2)$, by application of the rule;
$= (c^2 + d^2)(c + d)(c - d)$, by application of the rule to $c^2 - d^2$.

2. Factor $81a^8 - 16y^{12}$.

Solution. $81a^8 - 16y^{12} = (9a^4 + 4y^6)(9a^4 - 4y^6)$
$= (9a^4 + 4y^6)(3a^2 + 2y^3)(3a^2 - 2y^3)$.

ORAL EXERCISES

Factor:

1. $c^2 - d^2$.
2. $a^2 - 4$.
3. $h^2 - 1$.
4. $4m^2 - 1$.
5. $9 - r^2$.
6. $1 - k^2$.
7. $1 - 4k^2$.
8. $4x^2 - 25$.
9. $9x^2 - 4y^2$.
10. $16x^4 - y^2$.
11. $9c^2 - 64d^2$.
12. $100a^2 - 121b^2$.
13. $121x^2 - 49y^2$.
14. $144c^2 - 121d^2$.
15. $169a^2 - 100b^2$.
16. $25x^4 - 36y^4$.
17. $36x^4 - 49y^8$.
18. $16x^4 - 121a^2$.
19. $64x^4 - 81z^2$.
20. $100x^2 - 121y^4$.
21. $144x^6 - 169y^8$.

Solve for x,

22. $x(a+3) = a^2 - 9$.
23. $x(c-5) = c^2 - 25$.
24. $x(k+7) = k^2 - 49$.
25. $x(2+c) = 4 - c^2$.

EXERCISES

Factor the following:

1. $a^4 - b^4$.
2. $x^4 - 1$.
3. $a^4 - 16$.
4. $r^4 - 81$.
5. $16a^4 - 1$.
6. $81 - c^4$.
7. $625x^4 - 1$.
8. $625 - a^4$.
9. $625x^4 - 16y^4$.
10. $a^{12} - c^{16}$.
11. $c^4d^4 - 1$.
12. $c^8 - 81$.
13. $c^8 - d^8$.
14. $a^4b^8 - 16$.
15. $x^4y^8 - z^4$.
16. $(x+y)^2 - z^2$.

Hint. $(x+y)^2 - z^2 = [(x+y)+z][(x+y)-z]$ etc.

17. $(x-y)^2 - 25$.
18. $(2x+5)^2 - 4y^2$.
19. $(3a-7)^2 - x^2$.
20. $4(x-9)^2 - 9a^2$.
21. $9(a-b)^2 - 25c^2$.
22. $25x^2(c+d)^2 - 9y^2$.
23. $4a^2 - (2c-d)^2$.

Hint. $4a^2 - (2c-d)^2 = [2a + (2c-d)][2a - (2c-d)]$ etc.

24. $25x^2 - (a-3b)^2$.
25. $49a^2 - (3x+2)^2$.
26. $81a^2 - 4(c^4+2)^2$.
27. $100 - a^2(b+c)^2$.

28. $121a^2 - 4b^2(c - 3d^2)^2$.

29. $y^2z^2 - (y + z)^2$.

30. $c^2e^2 - (c - e)^2$.

31. $c^2d^2 - (c^2 - d)^2$.

32. $9a^2b^2 - 4(a - b)^2$.

33. $25a^2b^2 - 9(a + b)^2$.

34. $4c^2d^2 - (c - 2d)^2$.

35. $16c^2d^2 - (a^2 + x)^2$.

36. $25c^2d^2 - 3^2(c + d)^2$.

37. $49r^2s^2 - 9(r^2 - s^2)^2$.

Some polynomials of four or six terms may be arranged as the difference of two squares and factored as in the preceding exercises.

EXERCISES

Factor:

1. $x^2 + 2xy + y^2 - z^2$.

Solution.
$$x^2 + 2xy + y^2 - z^2 =$$
$$(x + y)^2 - z^2 =$$
$$(x + y + z)(x + y - z).$$

2. $x^2 + 2x + 1 - y^2$.

3. $a^2 - 6a + 9 - b^2$.

4. $c^2 - 10c + 25 - 4d^2$.

5. $4a^2 + 4a + 1 - 9b^2$.

6. $16 + 24a + 9a^2 - 25b^2c^2$.

7. $4a^2 - 12ab + 9b^2 - 9b^4$.

8. $25x^2 + 4y^2 - 20xy - 16z^2$.

9. $9x^2 + 25y^4 - 81z^4 - 30xy^2$.

10. $60ab - 25c^4 + 9b^2 + 100a^2$.

11. $1 - 14ab^2 + 49a^2b^4 - b^4$.

12. $4 - 20ab^2c^3 + 25a^2b^4c^6 - 4c^8$.

13. $9x^2 - 30xy + 25y^2 - 16z^4$.

14. $16x^6 - 8x^3y^2 + y^4 - z^4$.

15. $l^2 - m^2 + 2mn - n^2$.

Solution.
$$l^2 - m^2 + 2mn - n^2 =$$
$$l^2 - (m^2 - 2mn + n^2) =$$
$$l^2 - (m - n)^2 =$$
$$[l + (m - n)][l - (m - n)] =$$
$$(l + m - n)(l - m + n).$$

16. $x^2 - 4y^2 + 4yz - z^2$.

17. $x^2 - y^2 + 10yz - 25z^2$.

18. $c^2 - 4a^2 - 12ab - 9b^2$.

19. $4c^2 - a^2 + 10ab - 25b^2$

20. $9d^2 - a^2 + 6ab - 9b^2$.

21. $16e^2 - 25m^2 + 10mn - n^2$.

REVIEW EXERCISES

Factor:

1. $ax^2 - ay^2$.
2. $a^2c - 4b^2c$.
3. $a^3 + 6a^2b + 9ab^2$.
4. $c^6 + 2c^4 + c^2$.
5. $dx^2 - 25d^3$.
6. $25e^4 - 30e^3 + 9e^2$.
7. $2x^3 - 18xy^2$.
8. $r^7 + 2r^5 + r^3$.
9. $a^3b - 6a^2b^3 + 9ab^5$.
10. $5a^4c^2 - 45c^4$.
11. $a^3x^3 - 2a^2x^2 + ax$.
12. $ax^5 - a^5x$.
13. $2x^5 - 162x$.
14. $x^5y - 16xy^5$.
15. $5a^4 - 5$.
16. $x^5 - xy^4$.
17. $32x^4 - 1250$.
18. $3z(x+y)^2 - 12z^3$.
19. $20x^2 + 20x^3 + 5x^4 + 5x^5$.
20. $8r^4 - 8r^2(3s+t)^2$.
21. $8a^4 - 12a^2bc - 18a^2b^2 - 2a^2c^2$.
22. Solve for x, $x(a+2) = a^2 - 4$.
23. Solve for y, $b^2y + 5y = b^4 - 25$.
24. Solve for z, $z(c^2 + 25)(c-5) = c^4 - 625$.
25. Solve for m, $mh^2 - mhk = h^3 - hk^2$.
26. Solve for n, $acn + 2c - a^2c = 2cn - 2c$.

63. The quadratic trinomial. The type form is

$$x^2 + bx + c.$$

For many trinomials of this type two binomial factors may be found of the form $(x+r)(x+s)$. The method of factoring to be used is the reverse of the method of

multiplying two binomials given on page 109. From a study of the four examples there given it is evident that

(1) *The first term of each binomial factor is the square root of the first term of the trinomial.*

(2) *The second terms of the binomials are those factors of the last term of the trinomial whose algebraic sum equals the coefficient of the middle term of the trinomial.*

EXAMPLES

1. Factor $x^2 + 12x + 32$.

Solution. $$x^2 + 12x + 32 = (x + ?)(x + ?).$$

It is necessary to find two numbers whose product is $+32$ and whose sum is $+12$.

Now $$32 = 1 \cdot 32 = 2 \cdot 16 = 4 \cdot 8.$$

The first two pairs are rejected, for each fails to give the sum $+12$. The third pair of factors of 32, namely 4 and 8, gives the correct sum.

Therefore $$x^2 + 12x + 32 = (x + 8)(x + 4).$$

2. Factor $a^2 - 11a + 24$.

Solution. Since 24 is positive, its two factors must have the same sign; since -11 is negative, both factors must be negative. Now $24 = 1 \cdot 24 = 2 \cdot 12 = 3 \cdot 8 = 4 \cdot 6$. By inspection of these products, -3 and -8 are found to be the required numbers.

Therefore $$a^2 - 11a + 24 = (a - 3)(a - 8).$$

3. Factor $c^2 - c - 42$.

Solution. The product -42 is negative; hence the required factors have *unlike* signs. The sum, -1, being negative, the negative factor of -42 must have the greater absolute value. Now $42 = 1 \cdot 42 = 2 \cdot 21 = 3 \cdot 14 = 6 \cdot 7$. We see that $6 + (-7) = -1$.

Therefore $$c^2 - c - 42 = (c + 6)(c - 7).$$

EXERCISES

Factor:

1. a^2+5a+6.
2. $c^2+7c+12$.
3. x^2+4x+4.
4. $x^2-8x+12$.
5. $m^2-13m+30$.
6. $h^2-16h+15$.
7. $n^2-8n+16$.
8. $k^2-17k+30$.
9. $m^2-10m+25$.
10. $r^2-5r-14$.
11. $s^2+2s-35$.
12. s^2-6s+9.
13. t^2+t-42.
14. $x^2+5x-14$.
15. $10a-39+a^2$.
16. a^4-4a^2-5.
17. a^4-14a^2+49.
18. $-x^2+x+20$.

Solution. By taking out the factor -1, we may factor as usual.

$$-x^2+x+20=-1(x^2-x-20)=-1(x-5)(x+4)$$
$$=(5-x)(x+4).$$

Note that (-1) is not retained as a factor in the final result.

19. $-x^2-x+12$.
20. $-m^2-8m+9$.
21. $-2a+63-a^2$.
22. $20-m-m^2$.
23. $24+2n-n^2$.
24. $80-2r-r^2$.
25. x^2-x-72.
26. $a^2-a-110$.
27. $a^2+12a+36$.
28. $c^2-50+5c$.
29. $h^2-55+6h$.
30. $-4s-21+s^2$.
31. $27+6x-x^2$.
32. $x^2+2rx-3r^2$.
33. $m^2+2mn-99n^2$.
34. $h^2-3hk-130k^2$.
35. $k^2-4kl+4l^2$.
36. $r^2+15rs-100s^2$.
37. $5hk+k^2-36h^2$.
38. $r^2s^2-3rst-40t^2$.
39. $a^2b^2+8abc+7c^2$.
40. $33p^2-m^2n^2-8mnp$.

REVIEW EXERCISES

Factor the following:

1. $x^4+7x^3+12x^2$.
2. $a^4-12a^3+36a^2$.
3. $2x^2+10x-48$.
4. $5c^3+15c^2-140c$.
5. $3m^4+66m^3+363m^2$.
6. $16a^3-4ab^2$.
7. $20c^5-5cd^4$.
8. $5x^3-20xy^2$.
9. $r^3s+10r^2s^2+21rs^3$.
10. $2h^3k^2+2h^2k^3-12hk^4$.

11. $2r^5 - 162r$.

12. $45m^3n^2 - 20mn^4$.

13. $2x^3y + 10x^2y^2 + 12xy^3$.

14. $4c^4d + 4c^3d^2 - 24c^2d^3$.

15. $5r^4s - 40r^3s^2 + 60r^2s^3$.

16. $39d^4 - 10cd^3 - c^2d^2$.

17. $18x^5 + 7kx^4 - k^2x^3$

18. $8abc^5 + a^2b^2c^4 - 65c^6$.

19. $6ah^2 + 3h^2c + 18ahk + 9chk$.

20. $30m^2nr - 10mr + 45m^2n - 15m$.

21. $2a^3 + 4a^2b + 2ab^2 - 2ac^2$.

22. $3c^3 + 6c^2d + 3cd^2 - 12c$.

23. $4x^3 - 4xy^2 - 8xyz - 4xz^2$.

24. Solve for x, $x(a+2) = a^2 + 5a + 6$.

25. Solve for y, $ym - y = m^2 - 4m + 3$.

26. Solve for z, $rz + r = r^2 + 5z - 20$.

27. Solve for r, $ar + 3ac - a^2 = 2cr + 2c^2$.

28. Solve for s, $2as + a^3 + a^4 = 6a^2 + a^2s$.

64. The general quadratic trinomial. The type form is

$$\boldsymbol{ax^2 + bx + c}.$$

For many trinomials of this type, two binomial factors of the form $(hx + k)(mx + n)$ may be found. The method of factoring such trinomials is illustrated in the following:

EXAMPLES

1. Factor $2x^2 + 5x + 3$.

Solution. $2x^2 + 5x + 3 = (?x + ?)(?x + ?)$.

To get the proper factors we must supply such numbers for the interrogation points in (1) and in (2) as will give (3).

$$\begin{array}{rrl} ?x + ? & & (1) \\ ?x + ? & & (2) \\ \hline 2x^2 + ?x & & \\ + ?x + 3 & & \\ \hline 2x^2 + 5x + 3 & & (3) \end{array}$$

$2x^2$ for the product of the first two terms of the binomials,
$+3$ for the product of the last two terms of the binomials, and
$+5x$ for the sum of the cross products.

Now $$2x^2 = 2x \cdot x,$$

and $$+3 = 1 \cdot 3.$$

The factors 1 and 3 may be substituted for the interrogation points in (1) and (2) in either of the following ways:

$$\begin{array}{r} 2x+1 \\ x+3 \end{array} \text{ (Incorrect)} \qquad \begin{array}{r} 2x+3 \\ x+1 \end{array} \text{ (Correct)}$$

The first pair is rejected, for it fails to give a product having the required middle term, $+5x$. The second pair gives the correct product.

Therefore $\quad 2x^2+5x+3=(2x+3)(x+1).$

2. Factor $3x^2-x-10$.

Solution. $3x^2=3x\cdot x$

$$-10=1\cdot -10=-1\cdot 10=2\cdot -5=-2\cdot 5.$$

Test the following pairs of binomials:

$$\begin{array}{cccccccc} 3x+1 & x+1 & 3x-1 & x-1 & 3x+2 & x+2 & 3x-2 & x-2 \\ x-10 & 3x-10 & x+10 & 3x+10 & x-5 & 3x-5 & x+5 & 3x+5 \end{array}$$

Only the last pair gives the desired product.

Therefore $\quad 3x^2-x-10=(x-2)(3x+5).$

After a little practice it will usually be found unnecessary to write down all of the pairs of binomials that do not produce the required product.

If none of the pairs gives the required product, the given trinomial is prime.

EXERCISES

Factor:

1. $3x^2+7x+2$.

2. $2x^2+7x+6$.

3. $3x^2+8x+5$.

4. $2a^2-9a+10$.

Hint. Since the last term is positive and the second negative, only negative factors of 10 need be considered.

5. $5a^2-2a-3$.

6. $6a^2+7a-5$.

7. $10x^2+13x-3$.

8. $6c^2+7c-20$.

9. $10x^2+9x+2$

10. $6x^2+5x-6$.

11. $2x^2+13x+18$.

12. $3d^2-10d-25$.

13. $5x^2-38x-16$.

14. $10x^2+7x-6$.

15. $4k^2+20k+21$.

16. $9x^2 + 3x - 2.$
17. $16c^2 - 8c - 3.$
18. $10l^2 - 7l - 12.$
19. $12x^2 - 8x - 15.$
20. $14r^2 - 39r + 10.$
21. $21x^2 - 61x - 30.$
22. $25s^2 - 15s + 2.$
23. $36x^2 - 36x + 5.$
24. $36a^2 + 23a - 3.$
25. $49x^2 - 21x + 2.$
26. $2 + x - 15x^2.$
27. $12x^4 + x^2 - 20.$

HINT. $12x^4 + x^2 - 20 =$
$12(x^2)^2 + x^2 - 20.$

28. $50x^4 + 5x^2 - 3.$
29. $6 + 7x^2 - 5x^4.$
30. $12 + 17x^3 + 6x^6.$
31. $a^2 - 3ab + 2b^2.$
32. $a^2 + 2ab - 8b^2.$
33. $c^2 - cd - 12d^2.$
34. $2x^2 + 5xy + 2y^2.$
35. $2a^2 - 5ab + 2b^2.$
36. $3x^2 - 10xy + 3y^2.$
37. $10x^2 - 27xy + 5y^2.$
38. $12x^2 + 23xy - 2y^2.$
39. $30x^2 - 13xy - y^2.$
40. $30x^2 + 109xy + 30y^2.$

REVIEW EXERCISES

Factor the following:

1. $4x^2 + 10x + 4.$
2. $3x^3 + 18x^2 + 27x.$
3. $20x^4 - 60x^3 + 45x^2.$
4. $50ax^2 - 140ax + 98a.$
5. $27x^3 - 36x^4 + 12x^5.$
6. $8x^4 + 2x^2 - 1.$
7. $60x^3 - 35x^2 - 60x.$
8. $3x^3y - 4x^2y^2 + xy^3.$
9. $12x^5y + 21x^3y^3 - 6xy^5.$
10. $5x^3y + 10x^2y^2 - 75xy^3.$
11. $6x^2r^2 + 12ar^2x + 4rx^2 + 8axr.$
12. $-18ax^2y - 45a^2xy - 6ax^2 - 15a^2x.$
13. $45x^4y^2 + 21x^3y - 6x^2.$
14. Solve for r, $r(a+1) = 2a^2 + 3a + 1.$
15. Solve for s, $sb + sc = b^2 - bc - 2c^2.$
16. Solve for x, $2hx = 6h^2 + h - 2 + x.$
17. Solve for y, $cy - cd - 2c^2 = dy - 3d^2.$

65. A binomial the sum or the difference of two cubes. The type form is

$$a^3 \pm b^3.$$

a^3+b^3 divided by $(a+b)$ gives the quotient a^2-ab+b^2, and a^3-b^3 divided by $(a-b)$ gives the quotient a^2+ab+b^2.

Therefore $$a^3+b^3=(a+b)(a^2-ab+b^2), \tag{1}$$

and $$a^3-b^3=(a-b)(a^2+ab+b^2). \tag{2}$$

Formulas (1) and (2) above may be expressed in words as follows:

(1) *The factors of the sum of the cubes of two terms are (a) a binomial which is the sum of the terms and (b) a trinomial in which the terms are the square of the first term minus the product of the first and the second term plus the square of the second term.*

(2) *The factors of the difference of the cubes of two terms are (a) a binomial which is the difference of the terms and (b) a trinomial in which the terms are the square of the first term plus the product of the first and the second term plus the square of the second term.*

EXAMPLES

1. Factor a^3+27.

Solution. $$a^3+27=a^3+3^3=(a+3)(a^2-a\cdot 3+3^2)$$
$$=(a+3)(a^2-3a+9).$$

2. Factor $27a^6-64b^3$.

Solution. $$27a^6-64b^3=(3a^2)^3-(4b)^3$$
$$=(3a^2-4b)[(3a^2)^2+(3a^2)(4b)+(4b)^2]$$
$$=(3a^2-4b)(9a^4+12a^2b+16b^2).$$

EXERCISES

Factor the following:

1. $x^3 + y^3$.
2. $m^3 - n^3$.
3. $a^3 + 8$.
4. $b^3 - 64$.
5. $c^3 + 8d^3$.
6. $m^3 - 125n^3$.
7. $8 - x^3$.
8. $64 + y^3$.
9. $27 + 8a^3$.
10. $m^9 - n^3$.
11. $125x^3 - y^3$.
12. $1 - 27m^3n^9$.
13. $8x^3y^6 + 1$.
14. $216 - x^3$.
15. $m^6 + n^6$.
16. $x^6 + y^6$.
17. $x^{12} + y^{12}$.
18. $64x^6 + y^6$.
19. $a^6 + 64$.
20. $m^6 - n^6$.

HINT. This expression may be regarded either as the difference of two cubes, $(m^2)^3 - (n^2)^3$, or as the difference of two squares, $(m^3)^2 - (n^3)^2$. Since the difference of two squares is one of the simplest type forms to factor, one should always use it when possible.

Thus $$m^6 - n^6 = (m^3 + n^3)(m^3 - n^3) \text{ etc.}$$

21. $x^6 - y^6$.
22. $a^6 - 64$.
23. $x^6 - 64y^6$.
24. $1 - a^6$.
25. $64x^6 - 1$.
26. $a^{12} - b^6$.
27. $x^{12} - 1$.
28. $y^{12} - x^6$

REVIEW EXERCISES

1. $x^4 + xy^3$.
2. $2x^4y - 2xy^4$.
3. $2a^4 - 54a$.
4. $5x^6 - 40x^3$.
5. $x^4 - 2x^3 + 27x - 54$.
6. $x^6 - 7x^3 - 8$.
7. $x^{10} - 64x^4$.
8. $3x^4 + 2x^3 - 24x - 16$.
9. Solve for x, $3x - ax = 27 - a^3$.
10. Solve for x, $bx - 8 = b^3 - 2x$.
11. Solve for x, $c^2x - 27 = c^3 + 3cx - 9x$.
12. Solve for r, $a^3 + a^2r = 8 - 2ar - 4r$.
13. Solve for s, $abs + as + a = ab^3 - ab^2s$.

66. General directions for factoring. Since no general method of factoring can be stated in a few simple rules, the process must be learned by means of such type forms and typical solutions as are given in the preceding pages. When once these have been thoroughly mastered, readiness in factoring expressions which are represented by them becomes a matter of experience. Usually a student finds it comparatively easy to factor a list of exercises classified under a particular type form, yet a list of miscellaneous exercises he finds difficult. This usually indicates inability to determine the type of an expression from its appearance. Until the student, by careful study of the type forms, has acquired the ability to do this, he will make little progress. There are many types that are not included in this book, which the student who continues the study of algebra will meet later.

The following suggestions will prove helpful in solving the types here considered:

I. First look for a common monomial factor, and if there is one (other than 1), separate the expression into its greatest monomial factor and the corresponding polynomial factor.

II. Then by the form of the polynomial factor determine with which of the following types it should be classed and use the methods of factoring applicable to that type.

1. $ax + ay + bx + by$.	4. $x^2 + bx + c$.
2. $a^2 \pm 2ab + b^2$.	5. $ax^2 + bx + c$.
3. $a^2 - b^2$.	6. $a^3 \pm b^3$.

III. Proceed again as in II with each polynomial factor obtained until the original expression has been separated into its prime factors.

MISCELLANEOUS EXERCISES

Factor:

1. $x^3 - 4x$.
2. $2a^3 + 4a^2b + 2ab^2$.
3. $ax^3 + ax^2y + a^2x^2 + a^2xy$.
4. $4x^4 + 400x^8 - 116x^6$.
5. $2cx^3 + 2cx^2 - 12cx$.
6. $5a^3 - 10a^2b - 75ab^2$.
7. $10x^4 - 5x^3 - 30x^2$.
8. $5x^{10} + 40x^7$.
9. $x^3y + 3x^3 + 8y + 24$.
10. $x^5 - 4x^3 - 8x^2 + 32$.
11. $16x^4 - 8x^3 - 2x + 1$.
12. $a^5 - 25a^3 - a^2 + 25$.
13. $-45a^3x^3 - 3a^4x^2 + 18a^5x$.
14. $9x^2 - 30axy^2 + 25a^2y^4$.
15. $x^{12} - 63x^6 - 64$.
16. $x^6 - 64y^{12}$.
17. $6x^4 - 7x^2y - 5y^2$.
18. $3 + 15x - 2ax - 10ax^2$.
19. $x^7 - 27x^4 - x^3 + 27$.
20. $a^3 + 3a^2b - ab^2 - 3b^3$.
21. $20x^4y + 38x^3y^2 - 30x^2y^3$.
22. $8ax^2 - 2a - 20x^2 + 5$.
23. $9a^2 + 4b^2 - 12ab - 4x^2y^2$.
24. $4x^2 - x^4y^4 - 4xy + y^2$.
25. $x^2 + y^2 - 4a^2 - 2xy$.
26. $x^5 - 4x^3y^2 + x^2y^3 - 4y^5$.
27. $x^2 + x^3 - 6x^4$.
28. $1 - x^2 + x^3 - x^5$.
29. $1 - 4x^2 + 8x^3 - 32x^5$.
30. $x^6 - 8x^4 + 16x^2$.
31. $2x^3y + 4x^2y + 2xy - 2xy^3$.
32. $8a^2y - 8ay^2 + 2y^3 - 2y$.
33. $4x^4 + 4x^3 - 4x^6 + x^2$.
34. $a^6 - 13a^4 + 36a^2$.
35. $x^8y^2 - 4x^6y^2 + 3x^4y^6 - 12x^2y^6$.
36. Solve for x, $ax + ad + bd = ac + bc - bx$.
37. Solve for x, $ax + bx + 3 = a + b + 3x$.
38. Solve for x, $10r - 25 - 5x = r^2 - rx$.
39. Solve for x, $4d^2 + 2dx - c^2 = -cx$.
40. Solve for z, $4r^2 + 9 + 3z = 2rz + 12r$.
41. Solve for x, $m + 42 - 7x = m^2 - mx$.
42. Solve for x, $a + 15 + 5x = 2a^2 - 2ax$.
43. Solve for x, $-3c - x = 2c^2 - 2cx - 2$.
44. Solve for x, $6r^2 - 12r + 4x = r^3 - r^2x + 4rx - 8$.
45. Solve for y, $a^3 - c^3 = ay - cy$.

CHAPTER XIV

SOLUTION OF EQUATIONS BY FACTORING

67. Simple equations. A **simple** or **linear equation** in one unknown is one which may be put in such a form that

(*a*) the unknown does not appear in any denominator;
(*b*) only the first power of the unknown is involved.

Thus $3x - 4 = 7$, $2n + 5 = 4n + 1$, $ax + b = 0$, are simple equations. $(x - 1)(x + 3) = (x + 4)(x - 6)$ is also a simple equation, since on multiplying out it becomes $x^2 + 2x - 3 = x^2 - 2x - 24$, from which, after transposition, we get $4x + 21 = 0$.

In the preceding chapters only simple equations have been considered.

68. Quadratic equations. A **quadratic equation** in one unknown is one that may be put in such a form that

(*a*) the unknown does not appear in any denominator;
(*b*) the second but no higher power of the unknown is involved.

Thus $x^2 + 4x - 5 = 0$, $3x^2 - 4 = 5x + 6$, $ax^2 + bx + c = 0$, are quadratic equations.

A quadratic equation is often called an equation of the *second degree.*

The term in a quadratic equation which does not involve the unknown is called the constant term.

69. Solution of equations. The methods of factoring given in Chapter XIII enable us to solve many quadratic

equations. In the solution of equations by factoring, use is made of the following

Principle. *If the product of two or more factors is zero, one of the factors must be zero.*

Two or more, or even all, of the factors *may* be zero, but the vanishing of one is *sufficient* to make the product zero.

Consider the equation, in factored form,

$$(x-2)(x-4)=0. \tag{1}$$

If this equation is to be solved, all of the numbers which satisfy it must be found. That is, we must find every value of x for which the product on the left of (1) is zero. If the product is to equal zero, the foregoing principle requires that one of the factors be zero. Hence any value of x which satisfies (1) must make either $x-2=0$ or $x-4=0$. Hence x must equal either 2 or 4. On substituting 2 for x in (1) we obtain $(2-2)(2-4)=0$, or $0\cdot(-2)=0$. Hence 2 is a root of (1). On substituting 4 for x in (1) we obtain $(4-2)(4-4)=0$, or $2\cdot 0=0$. Hence 4 is a root of (1). A moment's inspection makes it clear that 2 and 4 are the only roots of the equation.

ORAL EXERCISES

For what value of x is each of the following expressions equal to zero?

1. $x-2$. **3.** $x-8$. **5.** $3x-12$.

2. $x+5$. **4.** $x+10$. **6.** $2x+4$.

7. What is the value of 2×0? of $(-2)\times 0$? of 9×0? of 0×21?

8. Make a general statement which includes all the results of Exercise 7.

What is the value of:

9. $(x-1)(x-2)$ when $x=3$? 1? 0?
10. $(x-4)(x-3)$ when $x=3$? 2? 4?
11. $(x-3)(x-5)$ when $x=-1$? 3? 6?
12. $(2x+1)(3x-1)$ when $x=-1$? $\frac{1}{8}$? 0?
13. $x(x-1)$ when $x=0$? 1? 2?
14. $(x-1)(x-2)(x-3)$ when $x=1$? 2? 3?
15. $3x(x+2)$ when $x=-2$? 0? 2?

In Exercises 16–22, which of the numbers at the right of each equation is a root of that equation?

16. $(x-3)(x-1)=0.$ — 1, 2, 3.
17. $(x+2)(x-4)=0.$ — $-4, -2, 2, 4.$
18. $x(3x-2)=0.$ — 0, 1, 2.
19. $(2x+1)(x+3)=0.$ — $-\frac{1}{2}, -2, -3.$
20. $(x-1)(x-2)(x-3)=0.$ — 1, 2, 3, 4.
21. $x(x+1)(x-1)=0.$ — $-1, 0, 1, 2.$
22. $(x-1)^2(x-9)=0.$ — 1, 9, 10.

What are the roots of the equations in Exercises 23–28?

23. $(x-1)(x+1)=0.$
24. $(x+2)(x-3)=0.$
25. $(x-2)(x-1)(x+2)=0.$
26. $(2x-3)(x-1)=0.$
27. $(5x+2)(x+2)=0.$
28. $(2x-1)(3x+5)=0.$

29. Is there any one value of x which makes both factors of $(x-3)(x+6)$ equal to zero?

EXAMPLES

1. Solve the quadratic equation $x^2+5x=6$.

Solution. Transposing, $x^2+5x-6=0.$

Factoring, $(x-1)(x+6)=0.$

The value of x which makes the factor $x-1$ equal to zero is a root of the quadratic. Setting $x-1=0$, we obtain $x=1$.

Similarly, the value of x which makes $x + 6$ equal to zero is a root of the quadratic. Setting $x + 6 = 0$, we obtain $x = -6$.

Hence 1 and -6 are the roots of the given quadratic equation.

Check. Substituting 1 for x in $x^2 + 5x = 6$, we have $1 + 5 = 6$.
Substituting -6 for x in $x^2 + 5x = 6$, we have $36 - 30 = 6$.

2. Solve the quadratic equation $x^2 = 4x$.

Solution. Transposing, $x^2 - 4x = 0$.

Factoring, $x(x - 4) = 0$.

The factors are x and $x - 4$. The value which makes the first factor zero is $x = 0$. The value of x which makes the second factor zero is $x = 4$. Hence the roots of $x^2 - 4x = 0$ are 0 and 4.

Check. Substituting $x = 0$ in $x^2 = 4x$, $0 = 0$.
Substituting $x = 4$ in $x^2 = 4x$, $16 = 16$.

For solving an equation in one unknown by factoring we have the

Rule. *Transpose the terms so that the right member is zero. Then factor the expression on the left, set each factor which contains the unknown equal to zero, and solve the resulting equations.*

It must be kept in mind that a root of an equation is a number which satisfies the equation.

One should never divide each member of an equation by an expression containing the unknown, for in this manner roots may be lost.

Thus, if in Example 2 we had divided both sides of the equation by x, the resulting equation would have been $x - 4 = 0$, which, to be sure, gives us one root of the given equation. But we have lost the root $x = 0$ which corresponds to the factor by which we divided.

EXERCISES

Find the roots of the following quadratic equations, and check as directed by the teacher:

1. $x^2 - 9 = 0$.
2. $x^2 - 25 = 0$.
3. $x^2 = 16$.
4. $x^2 = 49$.
5. $x^2 - 2x = 0$.
6. $2x^2 + 6x = 0$.
7. $x^2 = 5x$.
8. $3x^2 = 9x$.
9. $x^2 - 4x + 3 = 0$.
10. $x^2 - 6x + 9 = 0$.
11. $x^2 - 7x = -12$.
12. $x^2 + x = 20$.
13. $x^2 + 8 = -6x$.
14. $x^2 = 3x + 10$.
15. $4x^2 - 16x = 0$.
16. $5x^2 + 35x = 0$.
17. $8 - 9x = -x^2$.
18. $12x - 28 = -x^2$.
19. $x^2 - 16x + 64 = 0$.
20. $x^2 - 54 = 15x$.
21. $12 - 25x + 12x^2 = 0$.
22. $3x^2 + x = 4$.
23. $18x^2 = 9x + 20$.
24. $(x + 8)(x + 1) = -12$.
25. $x^2 + 9x - 12 = 3x + 15$.
26. $x + 12 = x^2$.
27. $50x + 24 = 25x^2$.
28. $(x - 11)(x + 3) = 2x + 42$.
29. $x^2 - a^2 = 0$.
30. $x^2 = 4b^2$.
31. $x^2 = 16k^2$.
32. $x^2 - 2bx + b^2 = 0$.
33. $x^2 + 4a^2 = 4ax$.
34. $x^2 - ax = 0$.
35. $x^2 = 7ax$.
36. $2x^2 - 7x = 21 - 6x$.
37. $x^2 - bx - ax = 0$.
38. $x^2 - 5x = 2ax$.
39. $x^2 - ax - bx + ab = 0$.
40. $x^2 + bx = 4b + 4x$.

70. Cubic equations. An equation in x which may be put in the form

$$ax^3 + bx^2 + cx + d = 0,$$

where the coefficients a, b, c, and d represent numbers, is called a **cubic equation**, or an equation of the third degree.

Some equations of higher degree than the second may be solved by the method of factoring.

EXAMPLE

Solve the cubic equation $x^3 - 9x = 9 - x^2$. (1)

Solution. Transposing, $x^3 + x^2 - 9x - 9 = 0.$

Grouping, $x^2(x+1) - 9(x+1) = 0.$

Factoring, $(x+1)(x^2-9) = 0,$

or $(x+1)(x+3)(x-3) = 0.$

Setting each factor equal to zero,

$$x + 1 = 0, \text{ whence } x = -1,$$
$$x + 3 = 0, \text{ whence } x = -3,$$
$$x - 3 = 0, \text{ whence } x = 3.$$

Therefore -1, -3, and 3 are the roots of the equation $x^3 - 9x = 9 - x^2$.

Check. When $x = -1$, (1) becomes $-1 + 9 = 9 - 1.$
When $x = -3$, (1) becomes $-27 + 27 = 9 - 9.$
When $x = 3$, (1) becomes $27 - 27 = 9 - 9.$

EXERCISES

Find the roots of the following equations, and check:

1. $x^3 - 9x = 0.$
2. $3x^3 = 12x.$
3. $x^3 + 4x^2 - 12x = 0.$
4. $x^3 - 14x = 5x^2.$
5. $5x^2 = 2x - 3x^3.$
6. $x^3 - x^2 - 4x + 4 = 0.$
7. $2x^3 - x^2 = 8x - 4.$
8. $3x^3 - 2x^2 - 12x + 8 = 0.$
9. $x^3 - 25 = 25x - x^2.$
10. $2(x^3 - x) = 3(1 - x^2).$
11. $x^4 - 5x^2 + 4 = 0.$
12. $x^4 - 13x^2 + 36 = 0.$
13. $x^4 = 4x^2.$
14. $x^4 = 26x^2 - 25.$

PROBLEMS

1. The square of a certain number, plus the number itself, is 42. Find the number.

HINT. $n^2 + n = 42.$

2. If from the square of a certain number three times the number be taken, the remainder will be 54. Find the number.

3. If to the square of a certain number the sum of twice the number and 9 be added, the result will be 129. Find the number.

4. Three times the square of a certain number is equal to four times the number. What is the number?

5. A certain number is added to 16, and the same number is also added to 21; the product of the two sums is 546. What is the number?

6. A certain number is subtracted from 15, and the same number is also subtracted from 25; the product of the remainders is 119. Find the number.

7. From 30 a certain number is subtracted, and the same number is added to 18; the product of the results thus obtained is 560. Find the number.

8. If a certain number be added to 15, and the same number be subtracted from 22, the product of the sum and difference thus obtained will be 36 more than 50 times the number. Find the number.

9. If from the square of four times a certain number, five times the number be taken, the result will be 15 times the square of the number. Find the number.

The student has probably observed that a quadratic equation has two roots — one corresponding to each factor which contains an unknown. In Problems 1–9, where the question is asked about a "certain number," both roots of the equation are solutions of the problem, since both are numbers. In problems like the 10th, where the question is asked about some measurement, it frequently happens that one of the roots is a number which cannot measure the particular kind of thing with which the problem deals. Thus one root of the equation to which Problem 10 leads is -20; and -20 as the length of a lot is meaningless. Hence we reject it as a solution of the problem, in spite of the fact that it occurs as a root of the equation. Similarly, if a problem dealing with dimensions yields an equation with a negative root, or if a problem asking "how many men" yields an equation with a fractional root, we reject these

roots. Although they satisfy the algebraic conditions of the equation to which the problem leads, they fail to comply with the physical conditions of the problem itself, and consequently should not be retained as answers.

10. The depth of a certain lot whose area is 1600 square feet is four times its frontage. Find its dimensions.

11. The area of the floor of a certain room is 54 square yards. The length is 3 yards more than the breadth. What are the dimensions of the floor?

12. The area of a rectangular field is 216 square rods. The field is 6 rods longer than it is wide. Find its dimensions.

13. The sum of the squares of two consecutive integers is 313. Find the numbers.

14. The sum of the squares of two consecutive odd integers is 514. Find the numbers.

15. The sum of the squares of three consecutive odd integers is 251. Find the numbers.

16. An uncovered box 6 inches deep, with square bottom, has 112 square inches of inside surface. Find the other inside dimensions.

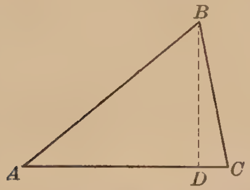
17. Remembering that the faces of a cube are squares, find the edge of a cubical box whose entire outer surface is 216 square inches.

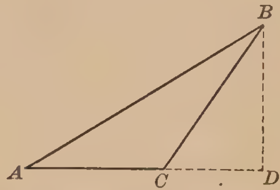
18. A rectangular box is three times as long and twice as wide as it is deep. There are 550 square inches in its entire outer surface. Find its dimensions.

19. A box is 3 inches longer and 1 inch wider than it is deep. There are 62 square inches in its entire outer surface. Find its dimensions.

The *altitude* of a triangle is the perpendicular from any vertex to the side opposite. This side is called the *base*.

In the adjacent figures BD is the altitude and AC is the base of each triangle.





If a is the altitude of a triangle and b its base, the area of the triangle is $\frac{ab}{2}$.

In making use of this and similar formulas the unit in terms of which the lines are measured must be specified.

20. The area of a triangle is 40 square feet; its altitude is 8 feet. Find the base.

21. The altitude of a triangle is twice the base and the area is 36 square feet. Find the base and the altitude.

22. The base of a triangle is 5 times the altitude and the area is 40 square feet. Find the base and the altitude.

23. The area of a triangle is 48 square inches; the base is six times the altitude. Find the altitude and the base.

24. The area of a triangle is 24 square feet; the altitude is 2 feet longer than the base. Find the altitude and the base.

HINT. Let $x =$ the base in feet.

Then $x + 2 =$ the altitude in feet,

and $\frac{x(x+2)}{2} = \frac{x^2 + 2x}{2} =$ the area.

Therefore $\frac{x^2 + 2x}{2} = 24.$

Multiplying each member by 2, this equation becomes

$$x^2 + 2x = 48.$$

25. The altitude of a triangle is 4 feet longer than the base. The area is 6 square feet. Find the base and the altitude.

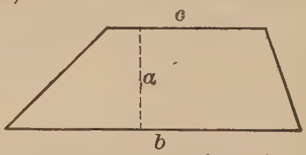
26. One leg of a right triangle is 4 yards longer than the other and the area is 30 square yards. Find the legs.

27. The area of a right triangle is 26 square yards and one leg is 8 feet longer than the other. Find the legs.

28. The area of a triangle is $2\frac{5}{8}$ square feet and the base is 6 inches longer than twice the altitude. Find the base and altitude.

29. The area of a triangle is 12 square yards and the altitude is 6 feet less than twice the base. Find the base and the altitude.

A *trapezoid* is a four-sided figure, two of whose sides are unequal and parallel.



The bases of a trapezoid are the two parallel sides, b and c.

The altitude, a, is the perpendicular distance between the bases.

The area of a trapezoid is given by the formula $\dfrac{a(b+c)}{2}$.

30. Find the area of a trapezoid whose bases are 10 and 18 and whose altitude is 12.

31. The altitude of a trapezoid is 8 inches, its area is 96 square inches, and one base is 4 inches longer than the other. Find each base.

HINT. Let $\quad x =$ the length of one base in inches.

Then $\quad x + 4 =$ the length of the other base in inches,

and $\dfrac{8(x+x+4)}{2}$, or $8x + 16 =$ area of the trapezoid.

Therefore $\quad 8x + 16 = 96.$

32. One base of a trapezoid is 12 feet, the other base is three times the altitude, and the area is 90 square feet. Find the altitude.

33. The altitude of a trapezoid is one half the shorter base and the latter is two thirds of the other base. The area is 250 square feet. Find the bases and the altitude.

34. One base of a trapezoid is 12 feet longer than the altitude, the other base is 6 feet longer than the altitude, and the area is 112 square feet. Find the bases and the altitude.

35. The bases of a trapezoid are respectively 1 foot and 5 feet longer than the altitude, and the area is 30 square yards. Find the bases and the altitude.

36. One base of a trapezoid is 6 feet longer than the other, the altitude is one half the sum of the bases, and the area is 9 square yards. Find the bases and the altitude.

37. The area of a trapezoid is 8 square yards, the altitude equals one base, and the other base exceeds the altitude by 2 feet. Find the bases and the altitude.

38. One base of a trapezoid exceeds the other by 10 feet, the altitude is 2 feet longer than five times the shorter base, and the area is 22 square yards. Find the altitude and the two bases.

CHAPTER XV

FRACTIONS

71. Algebraic fractions. The expression $\frac{a}{b}$, in which a and b represent numbers or polynomials, is an **algebraic** fraction. It is read "a divided by b," or "a over b." A fraction is an indicated quotient in which the dividend is the numerator and the divisor the denominator. The numerator and denominator are often called the *terms* of a fraction.

Certain operations upon fractions, such as multiplying both numerator and denominator by a number (raising to higher terms), and dividing both numerator and denominator by a number (reducing to lower terms), are often necessary before the processes of addition or subtraction of two or more fractions can be performed.

The change of a fraction to lower or to higher terms, and the addition and the subtraction of fractions in both arithmetic and algebra, depend on the

Principle. *The numerator and the denominator of a fraction may be multiplied by the same expression or divided by the same expression without changing the value of the fraction.*

Thus $\quad \frac{3}{4} = \frac{3 \cdot 4}{4 \cdot 4} = \frac{12}{16}$, and $\frac{18}{30} = \frac{18 \div 6}{30 \div 6} = \frac{3}{5}$.

Similarly, $\quad \frac{a}{b} = \frac{a \cdot n}{b \cdot n} = \frac{an}{bn}$, and $\frac{a}{b} = \frac{a \div n}{b \div n} = \frac{a/n}{b/n}$.

Since $\frac{4}{4}$, $\frac{6}{6}$, and $\frac{n}{n}$ are each equal to 1, each of the four preceding illustrations is really a multiplication or a division of a fraction by 1. This produces no change in the numerical value of *any* fraction, though it may change its form.

ORAL EXERCISES

Read the result of multiplying both numerator and denominator of each fraction by the factor on its right:

1. $\frac{1}{3}$. 2.   3. $\frac{3}{7}$. 5.   5. $\frac{5}{x}$. 4.   7. $\frac{2}{a+x}$. a.

2. $\frac{2}{5}$. 3.   4. $\frac{a}{z}$. 3.   6. $\frac{a}{b}$. x.   8. $\frac{a-x}{a+x}$. ax.

How are the fractions on the right obtained from those which precede?

9. $\frac{1}{2}, \frac{3}{6}$.   11. $\frac{2}{5}, \frac{8}{20}$.   13. $\frac{4}{9}, \frac{8a}{18a}$.

10. $\frac{1}{5}, \frac{3}{15}$.   12. $\frac{3}{7}, \frac{6}{14}$.   14. $\frac{7}{10x}, \frac{21x}{30x^2}$.

Read the result of dividing both numerator and denominator of each fraction by the number on its right:

15. $\frac{2}{4}$. 2.   17. $\frac{5}{20}$. 5.   19. $\frac{2a}{6}$. 2.   21. $\frac{2x}{x+xy}$. x.

16. $\frac{3}{9}$. 3.   18. $\frac{12}{18}$. 6.   20. $\frac{a}{ab}$. a.   22. $\frac{a+ab}{a-ab}$. a.

How are the fractions on the right obtained from those which precede?

23. $\frac{4}{10}, \frac{2}{5}$.   25. $\frac{24}{30}, \frac{4}{5}$.   27. $\frac{xy^2}{x^2y}, \frac{y}{x}$.

24. $\frac{12}{18}, \frac{2}{3}$.   26. $\frac{a^2}{ab}, \frac{a}{b}$.   28. $\frac{2x^3}{6x^2y}, \frac{x}{3y}$.

72. Reduction of fractions to lowest terms. A fraction is in its **lowest terms** when no factor except 1 is common to both numerator and denominator.

Cancellation is the process of dividing the numerator and the denominator of a fraction by a factor common to both.

EXAMPLES

Reduce to lowest terms:

1. $\dfrac{54\,a^3b^5c^2}{144\,ab^2c^4}$.

Solution. $$\frac{54\,a^3b^5c^2}{144\,ab^2c^4}=\frac{\cancel{2}\cdot\overset{3}{\cancel{3^3}}\cdot\overset{a^2b^3}{\cancel{a^3}\cancel{b^5}\cancel{c^2}}}{\underset{2^3}{\cancel{2^4}}\cdot\cancel{3^2}\cdot\underset{c^2}{\cancel{a}\cancel{b^2}\cancel{c^4}}}=\frac{3\,a^2b^3}{8\,c^2}.$$

2. $\dfrac{4\,a^8-324\,a^4}{6\,a^5-36\,a^3-162\,a}$.

Solution. $$\frac{4\,a^8-324\,a^4}{6\,a^5-36\,a^3-162\,a}=\frac{\overset{2\,a^3}{\cancel{4}\,\cancel{a^4}}(a^2+9)\cancel{(a+3)}\cancel{(a-3)}}{\underset{3}{\cancel{6}}\,\cancel{a}(a^2+3)\cancel{(a+3)}\cancel{(a-3)}}$$
$$=\frac{2\,a^3(a^2+9)}{3\,(a^2+3)}.$$

The pupil should note that a factor which occurs one or more times in both numerator and denominator of a fraction can be canceled only the same number of times.

For reducing a fraction to its lowest terms we have the

Rule. *Separate the numerator and the denominator into their prime factors and cancel the factors common to both.*

Cancellation as used in the rule means an actual division of the numerator and the denominator by the same expression. Therefore *only factors which are common to the numerator and the denominator can be canceled.*

The terms (the parts connected by plus or minus signs) in polynomial numerators and denominators, even if alike, can never be canceled. For example, it would be incorrect to "cancel" thus: $\dfrac{5+\cancel{2}}{6+\cancel{2}}$, as the resulting fraction would be $\frac{5}{6}$ instead of the true value, $\frac{7}{8}$. Similarly, in the fraction $\dfrac{x+a+4\,c^2}{y+a+8\,c^2}$ no cancellation is possible.

We have seen that we may multiply or divide both numerator and denominator of a fraction by the same number without affecting the value of the fraction. But we should never forget that *adding the same number to or subtracting the same number from both numerator and denominator changes the value of the fraction. Also, squaring both numerator and denominator leads to a different value.* Compare this statement with the operations that may be performed on each member of an equation as given on pages 39–40.

EXERCISES

Reduce to lowest terms:

1. $\dfrac{36a}{54a^2}$.
2. $\dfrac{52x^2}{130xy}$.
3. $\dfrac{84m^2}{126mn}$.
4. $\dfrac{x^3y}{xy^3}$.
5. $\dfrac{xy^2}{2x^2y}$.
6. $\dfrac{5ab^2c}{3ac^2}$.
7. $\dfrac{10ax^2}{25ax^3}$.
8. $\dfrac{27cd^2e}{45c^2e^3}$.
9. $\dfrac{108m^2n^3}{252md^2}$.
10. $\dfrac{2a}{a^2+ab}$.
11. $\dfrac{4c^2-8cd}{12c}$.
12. $\dfrac{10x^2y^2-5xy^3}{15xy^3}$.
13. $\dfrac{9a^2-6a}{3a-2}$.
14. $\dfrac{4x^2-25}{4x^2-20x+25}$.
15. $\dfrac{9-6x+x^2}{9-x^2}$.
16. $\dfrac{9x^2-1}{9x^2-9x+2}$.
17. $\dfrac{10a^2-2a}{15a^2+7a-2}$.
18. $\dfrac{c^3-4c}{c^5-8c^2}$.
19. $\dfrac{c^2-6c+8}{c^2+c-6}$.
20. $\dfrac{2x^3+x^2-3x}{3x^4-3x^2}$.
21. $\dfrac{50x-2x^3}{x^3+8x^2+15x}$.
22. $\dfrac{x^3-49x}{42x^2-27x^3+3x^4}$.
23. $\dfrac{a^3-a^2+2a-2}{3a^3-a^2+6a-2}$.
24. $\dfrac{2a^3y+2a^2xy+2acy+2cxy}{2ax+2a^2-ay-xy}$.
25. $\dfrac{m^2-n^2}{m^3-n^3}$.
26. $\dfrac{x^4-8x}{x^3-2x^2}$.
27. $\dfrac{a^2-4}{(a-2)^2}$.
28. $\dfrac{a^6-1}{a^2-1}$.
29. $\dfrac{a^4-x^4}{a^4+3a^2x^2+2x^4}$.
30. $\dfrac{27x^3+a^3}{9x^2-a^2}$.

73. Lowest common multiple. The **lowest common multiple** (L.C.M.) of two or more arithmetical or algebraic expressions is the expression having the least number of factors which will exactly contain each of the given expressions.

If two or more polynomials have no common factor other than 1, they are said to be **prime** to each other.

Thus $5a^2b$ and $4x^2y$ are prime to each other, as also are $2x^2+4x$ and x^2-9. On the other hand, x^2-9 and x^2-6x+9 are not prime to each other, since they contain the common factor $x-3$.

ORAL EXERCISES

Without separating the expressions into their prime factors, find the L.C.M. of the following:

1. 4, 6. **6.** 8, 12. **11.** 5, 10, 15. **16.** 8, 10, 20.

2. 5, 10. **7.** 7, 14. **12.** 6, 10, 12. **17.** 8, 20, 40.

3. 6, 8. **8.** 10, 15. **13.** 6, 8, 12. **18.** 3, 12, 15.

4. 6, 9. **9.** 12, 18. **14.** 7, 14, 28. **19.** 4, 8, 16.

5. 8, 10. **10.** 4, 6, 8. **15.** 7, 14, 21. **20.** 6, 18, 24.

EXAMPLE

Find the L.C.M. of $36a^2b^5$, $72a^3b^4$, and $108a^3b^2c^3$.

Solution.
$$36a^2b^5 = 2^2 \cdot 3^2 \cdot a^2b^5,$$
$$72a^3b^4 = 2^3 \cdot 3^2 \cdot a^3b^4,$$
$$108a^3b^2c^3 = 2^2 \cdot 3^3 \cdot a^3b^2c^3.$$

Since the L.C.M. must contain each of the expressions, it must have 2^3 as a factor. It will then contain 2^2, which occurs in both the first and the third monomials. Similarly, the L.C.M. must contain as factors 3^3, a^3, b^5, and c^3.

Therefore the L.C.M. is $2^3 \cdot 3^3 a^3b^5c^3$, which equals $216a^3b^5c^3$.

The method of finding the L.C.M. of two or more expressions is stated in the following

Rule. *Separate each expression into its prime factors. Then find the product of all the different prime factors, using each factor the greatest number of times it occurs in any one expression.*

EXERCISES

Find the L.C.M. of the following:

1. 96, 36, 40.

2. 45, 105, 175.

3. 50, 44, 110, 275.

4. a^2x, ax^3, a^2x^2.

5. mn, m^2, m^5n^2.

6. $2x$, $6x^2$, $4x^3$.

7. $3r$, 12, $15r^2$.

8. $4a$, $6a^2$, $8a^5$.

9. $2ax^2$, $3a^2x$, a^2cx^3.

10. $4x^2y$, $6x^3yz^2$, $10xz^2$.

11. $6a^2b$, $9ab^2c$, $30b^3c^3$.

12. $4c^2f$, $10df^2$, $24cd^2e^3$.

13. $2a^2c$, $18ax^2k^3$, $45x^3k$.

14. $x^2 - xy$, $2x^2y^3$.

15. $a^2 + ab$, $2ab$.

16. $5c - 10$, $10c$.

17. $b^2 - bk$, b^2k.

18. ac^2, $3a^2c$, $6a^2c - 9ac^2$.

19. $a^2 + ab$, $ab + b^2$.

20. $3x^3 + 6x^2y + 3xy^2$, $9x^3y - 9xy^3$, $6x^4 - 12x^3y + 6x^2y^2$.

Solution.

$$3x^3 + 6x^2y + 3xy^2 = 3x(x+y)^2$$
$$9x^3y - 9xy^3 = 3^2xy(x+y)(x-y)$$
$$6x^4 - 12x^3y + 6x^2y^2 = 2 \cdot 3x^2(x-y)^2.$$

Hence the L.C.M. is $2 \cdot 3^2x^2y(x+y)^2(x-y)^2$.

21. $a^2 - ab$, $2a - 2b$, $2a^2 - 2ab$.

22. $ax - ay$, $x^2 + xy$, $x^2 - y^2$.

23. $a^2 - 4$, $a^2 + 3a + 2$, $a^2 - a - 2$.

24. $4x^2 - 9$, $2x^2 + 7x + 6$, $2x^2 + x - 6$.

25. $x^2 - 1$, $x^3 - 1$, $x^2 + 2x + 1$.

26. $a^2 - 9$, $2a^3 + 6a^2 + 18a$, $2a^4 - 54a$.

27. $x^3 - 4x$, $4x^2 + 2x$, $2x^2 + 4x$, $2x^2 - 3x - 2$.

74. Equivalent fractions. Two fractions are equivalent when one can be obtained from the other either by multiplying or by dividing both numerator and denominator by the same expression.

For example, $\frac{3}{5}$ and $\frac{6}{10}$ are equivalent fractions; also $\frac{xy}{x^2}$ and $\frac{y}{x}$.

The **lowest common denominator** (L.C.D.) of two or more fractions is the L.C.M. of their denominators.

EXAMPLES

Reduce to respectively equivalent fractions having the lowest common denominator :

1. $\frac{3}{5}$, $\frac{5}{6}$, and $\frac{7}{20}$.

Solution. The L.C.M. of the denominators is 60. Multiplying the numerator and the denominator of the first fraction by 12, of the second by 10, and of the third by 3, we obtain $\frac{36}{60}$, $\frac{50}{60}$, and $\frac{21}{60}$ respectively.

2. $\dfrac{3x}{8yz^2}$ and $\dfrac{5y}{6x^2z}$.

Solution. The L.C.M. of the denominators is $24x^2yz^2$. Multiplying both numerator and denominator of the first fraction by the factor $3x^2$, which is a factor of the L.C.M., but not of the denominator of the fraction, gives $\dfrac{9x^3}{24x^2yz^2}$. Multiplying both numerator and denominator of the second fraction by the factor $4yz$, which is a factor of the L.C.M., but not of the denominator of this fraction, gives $\dfrac{20y^2z}{24x^2yz^2}$. Hence the required fractions are $\dfrac{9x^3}{24x^2yz^2}$ and $\dfrac{20y^2z}{24x^2yz^2}$.

3. $\dfrac{4a}{9b^2c}$ and $\dfrac{8b}{6ac^2}$.

Solution. The L.C.D. is $18ab^2c^2$.

Then $\dfrac{4a}{9b^2c}=\dfrac{4a\cdot 2ac}{9b^2c\cdot 2ac}=\dfrac{8a^2c}{18ab^2c^2}$, and $\dfrac{8b}{6ac^2}=\dfrac{8b\cdot 3b^2}{6ac^2\cdot 3b^2}=\dfrac{24b^3}{18ab^2c^2}$.

Therefore, to change two or more fractions (in their lowest terms) to respectively equivalent fractions having the L.C.D., we have the

Rule. *If the prime factors of the denominators are not apparent, rewrite the fractions with their denominators in factored form.*

Find the L.C.M. of the denominators of the fractions.

Multiply the numerator and the denominator of each fraction by those factors of this L.C.M. which are not found in the denominator of that fraction.

ORAL EXERCISES

State the lowest common denominator of the fractions in each of the following exercises. Then change the fractions to respectively equivalent fractions having this L.C.D.

1. $\frac{1}{2}, \frac{1}{3}$.	**6.** $\frac{1}{6}, \frac{1}{18}$.	**11.** $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.	**16.** $\frac{1}{2}, \frac{5}{6}, \frac{7}{12}$.
2. $\frac{1}{3}, \frac{1}{4}$.	**7.** $\frac{1}{10}, \frac{1}{15}$.	**12.** $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}$.	**17.** $\frac{1}{2}, \frac{3}{4}, \frac{3}{8}$.
3. $\frac{1}{3}, \frac{1}{6}$.	**8.** $\frac{1}{12}, \frac{1}{18}$.	**13.** $\frac{1}{4}, \frac{1}{8}, \frac{1}{12}$.	**18.** $\frac{2}{3}, \frac{3}{4}, \frac{1}{6}$.
4. $\frac{1}{4}, \frac{1}{8}$.	**9.** $\frac{1}{15}, \frac{1}{20}$.	**14.** $\frac{1}{10}, \frac{1}{15}, \frac{1}{20}$.	**19.** $\frac{2}{3}, \frac{5}{6}, \frac{7}{18}$.
5. $\frac{1}{6}, \frac{1}{10}$.	**10.** $\frac{1}{24}, \frac{1}{36}$.	**15.** $\frac{1}{2}, \frac{1}{4}, \frac{3}{8}$.	**20.** $\frac{4}{5}, \frac{9}{10}, \frac{4}{15}$.

EXERCISES

Change the following fractions to respectively equivalent fractions having the lowest common denominator:

1. $\frac{x}{3}, \frac{y}{4}, \frac{z}{5}$.

2. $\frac{3x}{8}, \frac{5x^2}{12}, \frac{7x^3}{24}$.

3. $\frac{5a}{7}, \frac{3ax}{14}, \frac{2ax^2}{21}$.

4. $\frac{3a^2}{4}, \frac{6ab}{10}, \frac{15a^2b}{24}$.

5. $\frac{3}{5a}, \frac{2c}{3d}, \frac{5a}{10d}$.

6. $\frac{3x}{4y}, \frac{5x}{6y}, \frac{3}{10y}$.

7. $\frac{2a+b}{3a}, \frac{a}{2a^2}, \frac{5a^2}{4a}$.

8. $\frac{c+d}{12c}, \frac{3c-d}{15c^2}, \frac{4a}{24c^3}$.

9. $\frac{7a}{cde^2}, \frac{5e}{cd^2e}, \frac{4c}{c^2d^2e^2}$.

10. $\frac{7x}{3xy^2}, \frac{4y}{5xz}, \frac{2z}{10yz^2}$.

11. $\dfrac{3}{x-5}, \dfrac{4}{x+5}.$

Solution. The L.C.D. is $(x-5)(x+5)$. Multiplying both numerator and denominator of the first fraction by $x+5$, and those of the second fraction by $x-5$, gives $\dfrac{3(x+5)}{(x-5)(x+5)}$ and $\dfrac{4(x-5)}{(x+5)(x-5)}$, or $\dfrac{3x+15}{x^2-25}$ and $\dfrac{4x-20}{x^2-25}$ respectively.

12. $\dfrac{5}{a-2}, \dfrac{7}{a+2}.$

13. $\dfrac{a}{a-b}, \dfrac{b}{a+b}.$

14. $\dfrac{x}{2x-5}, \dfrac{3x}{2x+5}.$

15. $\dfrac{-2c}{c-3}, \dfrac{3c}{c+3}.$

16. $\dfrac{2a}{4-a}, \dfrac{a}{4+a}, \dfrac{2}{a}.$

Solution. The L.C.D. is $(4-a)(4+a)a$. The respectively equivalent fractions are

$$\frac{2a(4+a)a}{(4-a)(4+a)a}, \quad \frac{a(4-a)a}{(4+a)(4-a)a}, \quad \text{and} \quad \frac{2(4-a)(4+a)}{a(4-a)(4+a)}.$$

17. $\dfrac{x}{x+y}, \dfrac{2x}{x-y}, \dfrac{2}{x}.$

18. $\dfrac{2a}{a-3}, \dfrac{3}{a+3}, \dfrac{-a}{2}.$

19. $\dfrac{5c}{c-5}, \dfrac{-2c}{c-5}, \dfrac{3}{c}.$

20. $\dfrac{m}{m-n}, \dfrac{-2m}{m-n}, \dfrac{2}{m}.$

21. $\dfrac{5}{a-2}, \dfrac{2}{3a-6}.$

Hint. Rewrite the fractions with the second denominator in factored form.

22. $\dfrac{a}{x^2-xy}, \dfrac{5}{x-y}.$

23. $\dfrac{3a}{a^2-b^2}, \dfrac{2a}{a+b}.$

24. $\dfrac{5a}{a^2-b^2}, \dfrac{-4a}{a-b}.$

25. $\dfrac{-3x}{x^2-y^2}, \dfrac{5x}{x-y}.$

26. $\dfrac{2x+3}{9-x^2}, \dfrac{3-2x}{9+3x}, \dfrac{x}{3-x}.$

27. $\dfrac{-2x}{x-2}, \dfrac{3}{x^2-4}, \dfrac{5x}{x^2-5x+6}.$

28. $\dfrac{2a-7}{a^3-9a}, \dfrac{3-5a}{a^3-5a^2+6a}, \dfrac{2}{a-3}.$

29. $\dfrac{a}{a^3-8}, \dfrac{2a}{a^3+2a^2+4a}.$

30. $\dfrac{5x}{2x^2+3x}, \dfrac{3x}{2x^2-x-6}.$

31. $\dfrac{-4x}{3-x}, \dfrac{5x}{3+8x-3x^2}.$

32. $\dfrac{-1}{x+y-3}, \dfrac{5x+y}{x^2+y^2-9+2xy}.$

NOTE. The problem of operating with fractions presented great difficulties to all the early races. The Egyptians and the Greeks, even down to the sixth century of our era, always reduced their fractions to the sum of several fractions, each of which had 1 for a numerator. For example, $\frac{5}{6}$ would be expressed as $\frac{1}{2}+\frac{1}{3}$. The Romans usually expressed all the fractions of a sum in terms of fractions with the common denominator 12. The Babylonians resorted to a similar device, but used 60 for the denominator. In some way they all attempted to evade the difficulty of considering changes in both numerator and denominator. The Hindus seem to have been the first to reduce fractions to a common denominator, though Euclid (300 B.C.) was familiar with the method of finding the least common multiple of two or more numbers.

75. Addition and subtraction of fractions. If two or more fractions have the same denominator, their sum is the fraction obtained by adding their numerators and writing the result over their common denominator.

For example, $\dfrac{2}{5}+\dfrac{4}{5}+\dfrac{7}{5}=\dfrac{13}{5}$, and $\dfrac{n}{d}+\dfrac{3n}{d}+\dfrac{5n}{d}=\dfrac{9n}{d}.$

If two fractions have the same denominators, their difference is the fraction obtained by subtracting the numerator of the subtrahend from the numerator of the minuend and writing the result over their common denominator.

For example, $\dfrac{7}{9}-\dfrac{5}{9}=\dfrac{2}{9}$, and $\dfrac{x}{d}-\dfrac{y}{d}=\dfrac{x-y}{d}.$

If it is required to add or to subtract two fractions having unlike denominators, the fractions must be changed

to respectively equivalent fractions having a common denominator; then their sum or their difference is obtained as explained above.

For example, to find the sum of $\frac{3}{4}+\frac{5}{8}+\frac{7}{10}$ we reduce the fractions to respectively equivalent fractions having the common denominator 40, by multiplying both numerator and denominator of $\frac{3}{4}$ by 10, of $\frac{5}{8}$ by 5, and of $\frac{7}{10}$ by 4. The fractions become $\frac{30}{40}$, $\frac{25}{40}$, and $\frac{28}{40}$ respectively, and their sum is $\frac{83}{40}$.

In adding algebraic fractions with unlike denominators, as $\frac{x}{y}$ and $\frac{n}{d}$, we proceed in a similar way.

Multiply both terms of $\frac{x}{y}$ by d, and of $\frac{n}{d}$ by y. The fractions become $\frac{xd}{yd}$ and $\frac{ny}{yd}$ respectively, the sum of which is $\frac{xd+ny}{yd}$. Similarly, $\frac{x}{y}-\frac{n}{d}=\frac{xd}{yd}-\frac{ny}{yd}$, which equals $\frac{xd-ny}{yd}$.

The pupil should always reduce fractional results to their lowest terms.

ORAL EXERCISES

Find the algebraic sum of:

1. $\frac{2}{3}+\frac{5}{3}$.
2. $\frac{7}{5}+\frac{3}{5}$.
3. $\frac{8}{9}+\frac{11}{9}$.
4. $\frac{13}{7}+\frac{6}{7}$.
5. $\frac{11}{5}+\frac{7}{5}$.
6. $\frac{14}{3}+\frac{11}{3}$.
7. $\frac{12}{15}+\frac{10}{15}$.
8. $\frac{17}{9}-\frac{1}{9}$.
9. $\frac{21}{a}-\frac{2}{a}$.
10. $\frac{3x}{y}-\frac{x}{y}$.
11. $\frac{a}{b}-\frac{c}{b}$.
12. $\frac{m}{n}-\frac{2m}{n}$.
13. $\frac{a}{2x}-\frac{c}{2x}$.
14. $\frac{5x}{y^3}-\frac{2a}{y^3}$.
15. $\frac{3x}{4x^2}+\frac{5}{4x^2}$.
16. $\frac{2a+c}{xy}-\frac{a}{xy}$.
17. $\frac{3r-s}{2r}-\frac{s-r}{2r}$.
18. $\frac{2x-y}{3x}-\frac{2y+x}{3x}$.
19. $\frac{x^2+y^2}{x+y}-\frac{y^2-x^2}{x+y}$.
20. $\frac{3x+5y}{x^2-y^2}-\frac{x-2y}{x^2-y^2}$.
21. $\frac{3}{a^2-b^2}-\frac{2}{a^2-b^2}$.

EXAMPLE

Simplify $\dfrac{5c}{2a^2} + \dfrac{5a-2c}{6ac} - \dfrac{2c-4}{3a}$.

Solution. The L.C.D. is $6a^2c$.

$$\frac{5c}{2a^2} + \frac{5a-2c}{6ac} - \frac{2c-4}{3a} = \frac{5c \cdot 3c}{2a^2 \cdot 3c} + \frac{(5a-2c)a}{6ac \cdot a} - \frac{(2c-4)2ac}{3a \cdot 2ac}$$

$$= \frac{15c^2 + (5a^2 - 2ac) - (4ac^2 - 8ac)}{6a^2c}$$

$$= \frac{15c^2 + 5a^2 - 2ac - 4ac^2 + 8ac}{6a^2c}$$

$$= \frac{5a^2 + 6ac - 4ac^2 + 15c^2}{6a^2c}.$$

Check. Setting the original expression equal to the final result and substituting 2 for a and 3 for c, we obtain

$$\frac{5c}{2a^2} + \frac{5a-2c}{6ac} - \frac{2c-4}{3a} = \frac{5a^2 + 6ac - 4ac^2 + 15c^2}{6a^2c}.$$

$$\frac{15}{8} + \frac{1}{9} - \frac{1}{3} = \frac{20 + 36 - 72 + 135}{72}.$$

$$\frac{135}{72} + \frac{8}{72} - \frac{24}{72} = \frac{119}{72}.$$

$$\frac{119}{72} = \frac{119}{72}.$$

Therefore, to find the algebraic sum of two or more fractions (in their lowest terms) we have the

Rule. *Reduce the fractions to respectively equivalent fractions having the lowest common denominator. Write in succession over the lowest common denominator the numerators of the equivalent fractions, inclosing each numerator in a parenthesis preceded by the sign of the corresponding fraction.*

Rewrite the fraction just obtained, removing the parentheses in the numerator.

Then combine like terms in the numerator and, if necessary, reduce the resulting fraction to its lowest terms.

EXERCISES

Find the algebraic sum of:

1. $\frac{3}{4}+\frac{5}{6}$.
2. $\frac{3}{10}+\frac{8}{15}$.
3. $\frac{9}{25}-\frac{3}{10}$.
4. $\frac{5}{12}-\frac{7}{18}$.
5. $\frac{3}{14}+\frac{2}{21}$.
6. $\frac{5}{6}-\frac{3}{16}$.
7. $\frac{7}{24}-\frac{15}{32}$.
8. $\frac{17}{40}+\frac{19}{48}$.
9. $\frac{15}{28}-\frac{87}{40}$.
10. $\frac{43}{25}+2\frac{29}{45}$.
11. $\frac{19}{24}-\frac{89}{54}$.
12. $\frac{55}{38}+\frac{43}{57}$.
13. $\frac{7}{6}-1\frac{4}{9}+3\frac{7}{12}$.
14. $\frac{11}{10}-\frac{13}{12}+2\frac{5}{16}$.
15. $\frac{13}{32}+1-3\frac{9}{48}$.
16. $\frac{31}{36}-\frac{41}{32}+\frac{11}{42}$.
17. $\frac{15}{26}-\frac{11}{39}-\frac{25}{52}$.
18. $\frac{23}{35}-\frac{19}{45}+\frac{27}{50}$.
19. $\frac{19}{48}-\frac{13}{72}+\frac{43}{36}$.
20. $\frac{11}{64}+2-\frac{17}{24}$.
21. $\frac{14}{69}-\frac{15}{46}+\frac{19}{115}$.
22. $\frac{3a}{2}+\frac{4a}{3}$.
23. $\frac{2a}{5}+\frac{3a}{2}+\frac{11a}{10}$.
24. $\frac{7a}{10}-\frac{6a}{15}+\frac{3a}{5}$.
25. $\frac{4x}{7}+\frac{2x}{21}-\frac{3x}{14}$.
26. $\frac{a+2}{8}-\frac{3a-2}{10}$.
27. $\frac{5x-3}{12}-\frac{2x-7}{14}$.
28. $\frac{3a-5}{4}-\frac{2-a}{6}+\frac{7a}{8}$.
29. $\frac{x-5a}{10}-\frac{3x+2a}{18}-\frac{a-7x}{20}$.
30. $\frac{4m-1}{6}-\frac{a-m}{8}-\frac{m+3a-5}{30}$.
31. $\frac{3}{x}+\frac{2}{3x^2}$.
32. $\frac{2}{3a}+\frac{5}{2a^2}-\frac{10a}{4a^3}$.
33. $\frac{2a}{5a^3}-\frac{3}{10a}-\frac{5}{6a^2}$.
34. $\frac{a}{x}+\frac{x}{a}-\frac{2}{ax}$.
35. $\frac{a}{c}-\frac{1}{ac}-\frac{3}{2a}$.
36. $\frac{7}{3a^2}-\frac{2}{ax}+\frac{3}{x^2}$.
37. $\frac{2a}{3x}+\frac{11}{2ax^2}-\frac{5x}{a}$.
38. $\frac{3a+1}{5a}-\frac{4a+3}{3a^2}-\frac{5-a}{6a}$.
39. $\frac{4x^2-5}{3x^2}-\frac{2-3x}{2x}+\frac{3x-7}{5x^3}$.
40. $\frac{2x}{5x^2y}-\frac{3y-1}{10xy^2}-\frac{4xy+3}{15x^2y^2}$.

41. $\dfrac{3a+2}{a^2-4}-\dfrac{2a-1}{a^2-3a-10}.$

Solution.

$$\frac{3a+2}{a^2-4}-\frac{2a-1}{a^2-3a-10}=\frac{3a+2}{(a+2)(a-2)}-\frac{2a-1}{(a+2)(a-5)}$$
$$=\frac{(3a+2)(a-5)}{(a+2)(a-2)(a-5)}-\frac{(2a-1)(a-2)}{(a+2)(a-5)(a-2)}$$
$$=\frac{(3a+2)(a-5)-(2a-1)(a-2)}{(a+2)(a-2)(a-5)}$$
$$=\frac{3a^2-13a-10-(2a^2-5a+2)}{(a+2)(a-2)(a-5)}$$
$$=\frac{3a^2-13a-10-2a^2+5a-2}{(a+2)(a-2)(a-5)}$$
$$=\frac{a^2-8a-12}{(a+2)(a-2)(a-5)}\text{ or }\frac{a^2-8a-12}{a^3-5a^2-4a+20}.$$

Unless other directions are given, the denominators should be retained in factored form throughout the process.

Check. Let $a=3$.

$$\frac{3a+2}{a^2-4}-\frac{2a-1}{a^2-3a-10}=\frac{a^2-8a-12}{(a+2)(a-2)(a-5)}.$$
$$\frac{9+2}{9-4}-\frac{6-1}{9-9-10}=\frac{9-24-12}{5\cdot1\cdot(-2)}.$$
$$\frac{11}{5}-\frac{5}{-10}=\frac{-27}{-10}\text{ or }\frac{27}{10}=\frac{27}{10}.$$

In checking work in fractions *such values must be chosen for the letters as will make no denominator zero.* This prevents the substitution of -2, 2, or 5 for a in the foregoing example.

42. $\dfrac{12}{x^2-9}-\dfrac{2}{x^2-5x+6}.$

43. $\dfrac{7}{a^2-49}-\dfrac{2}{a^2-6a-7}.$

44. $\dfrac{5}{4x^2-1}-\dfrac{3x}{8x^3-1}.$

45. $\dfrac{4c+3}{1-c^2}+\dfrac{2c}{c+c^2}.$

46. $\dfrac{2y+7}{3+y}-\dfrac{2y^2-35}{y^2-11y-42}.$

47. $\dfrac{5a+1}{a+3}-\dfrac{7}{2a}+\dfrac{2}{3}.$

48. $\dfrac{2}{x^2-7x}-\dfrac{3}{x}+\dfrac{3}{x-7}.$

49. $\dfrac{7a+b}{a^2-b^2}+\dfrac{2}{a-b}-\dfrac{3}{a+b}.$

50. $\dfrac{5c^2}{4c^2-1}-\dfrac{3c+2}{2c+1}+\dfrac{5c}{2c-1}$.

51. $\dfrac{m^3+m}{m^3-8}-\dfrac{3m+7}{m^3+2m^2+4m}-\dfrac{m-3}{m-2}$.

52. $\dfrac{a^2+2ab+b^2}{a^2-b^2}+\dfrac{ab}{a^2+ab}-\dfrac{2b}{ab-b^2}$.

53. $\dfrac{x^2+3}{x^2-x-2}-\dfrac{x}{x-2}+\dfrac{x}{x^2+x}$.

54. $\dfrac{a}{a^2-ab}+\dfrac{b+a}{a^2-2ab+b^2}+\dfrac{2b}{-ab+a^2}$.

55. $\dfrac{cx^2}{c^2-x^2}-\dfrac{2cx}{c^2-cx}-\dfrac{cx^2(c-2)}{c^3-cx^2}$.

56. $\dfrac{a+c}{a-c}-\dfrac{a^2-ac+c^2}{a^2+ac+c^2}-\dfrac{4a^2c}{a^3-c^3}$.

57. $\dfrac{x^2+3x+9}{x^2-3x+9}-\dfrac{x-3}{x+3}-\dfrac{54}{x^3+27}$.

58. $\dfrac{x-5}{2x+5}+\dfrac{-5x+x^2}{2x^2+15x+25}-\dfrac{2x-5}{x+5}$.

76. Changes of sign in a fraction. The *sign of a fraction* is the plus or minus sign placed before the line separating the numerator from the denominator. Hence there are in a fraction three signs to consider: the sign of the fraction, the sign of the numerator, and the sign of the denominator.

Now in division the quotient of two expressions having like signs is positive, and the quotient of two expressions having unlike signs is negative.

Therefore $+\dfrac{+8}{+2}=+4;\qquad +\dfrac{-8}{-2}=+4;$

$$-\frac{-8}{+2}=-(-4)=+4;\qquad -\frac{+8}{-2}=-(-4)=+4.$$

Or, in general terms, $+\dfrac{+a}{+b}=+\dfrac{-a}{-b}=-\dfrac{-a}{+b}=-\dfrac{+a}{-b}$.

These examples illustrate the

Principle. *Without altering the value of a fraction the following changes in sign may be made:*

(a) The sign of the numerator and the sign of the denominator.

(b) The sign of the numerator and the sign before the fraction.

(c) The sign of the denominator and the sign before the fraction.

Hence any fraction may be written in at least four ways, if proper changes of sign are made.

Thus $$+\frac{3x}{x-5}=\frac{-3x}{5-x}=-\frac{-3x}{x-5}=-\frac{3x}{5-x}.$$

Similarly,

$$+\frac{a-2b}{a-b+3c}=+\frac{2b-a}{b-a-3c}=-\frac{2b-a}{a-b+3c}=-\frac{a-2b}{b-a-3c}.$$

The pupil should note particularly that changing the sign of the numerator involves a change of sign in each term of the numerator. Similarly, a change of sign of the denominator involves a change of sign in each term of the denominator.

EXERCISES

Write each of the following fractions in three other ways:

1. $\frac{a}{-h}$.
2. $\frac{-c}{2d}$.
3. $-\frac{n}{d}$.
4. $-\frac{2}{a-c}$.
5. $\frac{-3}{x-y}$.
6. $\frac{x-2}{2x-3}$.
7. $-\frac{2a-b}{b^2-a^2}$.
8. $\frac{c-3d}{d^2-c^2}$.
9. $\frac{x-5}{5x-6-x^2}$.
10. $-\frac{2-x}{1-x^3}$.

77. Changing signs of factors of denominators. In their present form the L.C.D. of $\frac{a-3}{a-7}-\frac{2}{7-a}$ is apparently $(a-7)(7-a)$. But instead of taking these factors it is

better to apply the principle of section 76 and rewrite the fractions so that the denominators will be identical.

Thus $$\frac{a-3}{a-7}-\frac{2}{7-a}=\frac{a-3}{a-7}+\frac{2}{a-7}=\frac{a-1}{a-7}.$$

And $$\frac{x}{x-3}+\frac{4-2x}{3-x}=\frac{x}{x-3}+\frac{2x-4}{x-3}=\frac{3x-4}{x-3}.$$

Similarly, $$\frac{3}{x^2-4}-\frac{5}{2-x}=\frac{3}{(x+2)(x-2)}+\frac{5}{x-2}\text{ etc.}$$

In addition and subtraction of fractions whenever a factor occurs in one denominator which is the negative of a factor in another, the form of one of the fractions should be changed according to section 76. This will greatly simplify the work and decrease the likelihood of errors.

EXERCISES

Simplify the following:

1. $\dfrac{7}{x-5}+\dfrac{3}{5-x}$.

2. $\dfrac{a}{a-b}-\dfrac{b}{b-a}$.

3. $\dfrac{x^2-2a}{x-a}+\dfrac{a+x^2}{a-x}$.

4. $\dfrac{x-2}{3-x}-\dfrac{2+x}{x-3}+\dfrac{3x}{x-3}$.

5. $\dfrac{a}{5-a}+\dfrac{a^2+5a}{a^2-25}$.

HINT. $\dfrac{a}{5-a}+\dfrac{a^2+5a}{a^2-25}=\dfrac{-a}{a-5}+\dfrac{a^2+5a}{(a+5)(a-5)}$ etc.

6. $\dfrac{2}{c^2-9}-\dfrac{9}{3-c}$.

7. $\dfrac{3}{7-x}-\dfrac{4}{x^2-49}$.

8. $\dfrac{2a+5}{5a+1}+\dfrac{2a-5}{1-5a}-\dfrac{46a+1}{25a^2-1}$.

9. $\dfrac{a-b}{3a+2b}-\dfrac{10ab}{4b^2-9a^2}-\dfrac{a+b}{3a-2b}$.

10. $\dfrac{2x+1}{2x-1}-\dfrac{-6x+1}{1-4x^2}-\dfrac{2x-1}{2x+1}$.

11. $\dfrac{-11a+46-6a^2}{a^2+5a-14}+\dfrac{2a-5}{a+7}+\dfrac{7-4a}{2-a}$.

78. Reduction of a mixed expression to a fraction. The mixed number $3\frac{2}{5}$ really means $3 + \frac{2}{5}$. It is equal to $\frac{3}{1} + \frac{2}{5}$ or $\frac{15}{5} + \frac{2}{5} = \frac{17}{5}$. The corresponding case in algebra is represented by $a + \frac{b}{c}$.

Now $$a + \frac{b}{c} = \frac{a}{1} + \frac{b}{c} = \frac{ac}{c} + \frac{b}{c} = \frac{ac+b}{c}.$$

Similarly,

$$a + 1 + \frac{3a}{a+2} = \frac{a+1}{1} + \frac{3a}{a+2} = \frac{(a+1)(a+2)}{a+2} + \frac{3a}{a+2} = \frac{a^2+6a+2}{a+2}.$$

The process involved in this operation is really nothing more than the addition of two fractions, one of which has the denominator 1. Hence the method of section 75 applies.

ORAL EXERCISES

Reduce to fractional form:

1. $2 + \frac{1}{5}$.

2. $3 + \frac{1}{3}$.

3. $4 + \frac{1}{7}$.

4. $5 + \frac{2}{3}$.

5. $6 + \frac{5}{7}$.

6. $\frac{2}{3} + 3$.

7. $\frac{7}{3} + 3$.

8. $\frac{11}{2} - 4$.

9. $\frac{13}{5} - 3$.

10. $2 - \frac{4}{5}$.

11. $7 - \frac{6}{7}$.

12. $5 + \frac{1}{3}$.

13. $7 - \frac{3}{4}$.

14. $12 + \frac{2}{9}$.

15. $1 + \frac{a}{b}$.

16. $2 - \frac{a}{c}$.

17. $a - \frac{a}{x}$.

18. $c - \frac{x}{c}$.

19. $r - \frac{s^2}{r}$.

20. $s^2 - \frac{1}{4}$.

21. $s^2 - \frac{s^2}{4}$.

22. $r^2 - \left(\frac{r}{2}\right)^2$.

23. $\frac{s^2}{4} + s^2$.

24. $\left(\frac{s}{2}\right)^2 - s^2$.

25. $\left(\frac{r}{3}\right)^2 - \frac{-r^2}{9}$.

26. $\left(\frac{r}{5}\right)^2 - \frac{r^2}{5}$.

27. $2 + \frac{3}{r+1}$.

28. $5 - \frac{2}{s-1}$.

29. $x + \frac{1}{x+1}$.

30. $y - \frac{2}{y-2}$.

EXERCISES

Write the following mixed expressions in fractional form:

1. $a+\frac{2}{x}$.
2. $b-\frac{c}{d}$.
3. $2c+\frac{c}{a}$.
4. $5x-\frac{x}{d}$.
5. $\frac{m}{n}-2$.
6. $\frac{c}{5d}+3$.
7. $s+\frac{s}{2}$.
8. $x^2-\frac{3x^2}{5}$.
9. $s^2-\left(\frac{s}{2}\right)^2$.
10. $x^2-\left(\frac{2x}{3}\right)^2$.
11. $2a^2-\frac{2a^2}{3}$.
12. $a+2+\frac{a-2}{2}$.
13. $c-d+\frac{c^2-d^2}{c+d}$.
14. $x-2y-\frac{x^2+y^2}{x+2y}$.
15. $a^2+b^2-\frac{(a+b)^2}{2}$.
16. $5a-7-\frac{25a^2}{5a+7}$.
17. $m^2+mn+n^2-\frac{2m^3-n^3}{m-n}$.
18. $x^2-xy-\frac{x^3-y^3}{x+y}+y^2$.
19. $5-\frac{125-2x^3}{25+5x+x^2}-x$.
20. $\frac{x+2}{x-2}-3+\frac{x-3}{2x+3}$.
21. $x^3+4x-\frac{x^4-18}{x-2}+8+2x^2$.
22. $b+2-\frac{14b}{2b-1}-\left(2b-1-\frac{b^2+9b}{b+5}\right)$.

HINT. Removing parentheses and combining, we get

$$b+2-\frac{14b}{2b-1}-2b+1+\frac{b^2+9b}{b+5}=3-b-\frac{14b}{2b-1}+\frac{b^2+9b}{b+5} \text{ etc.}$$

23. $\left(2+\frac{6a}{b}\right)-\left(3-\frac{4a}{3b}\right)$.
24. $\left(3-\frac{7}{2ac}\right)-\left(2-\frac{5}{3c^2}\right)$.
25. $\left(x+\frac{4y^2}{x-2y}\right)-\left(\frac{x^2}{x+2y}+2y\right)$.

79. Multiplication of fractions. In algebra as in arithmetic the product of two or more fractions is the product of their numerators divided by the product of their denominators.

Thus $\frac{3}{4} \cdot \frac{5}{7} = \frac{15}{28}$.

Similarly, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$,

and $5 \cdot \frac{3}{7} = \frac{5}{1} \cdot \frac{3}{7} = \frac{15}{7}$.

In like manner $n \cdot \frac{a}{b} = \frac{n}{1} \cdot \frac{a}{b} = \frac{na}{b}$.

If a factor occurs one or more times in any numerator and in any denominator of the indicated product of two or more fractions, it should be canceled the same number of times from both, thus giving the product of the several fractions in lower terms.

Thus $\frac{a}{\cancel{b}} \cdot \frac{\cancel{b}}{c} = \frac{a}{c}$.

EXAMPLE

Multiply $\frac{8\,ac^2}{9\,d}$ by $\frac{15\,d^2e}{16\,a^2c^4}$.

Solution. $\frac{8\,ac^2}{9\,d} \cdot \frac{15\,d^2e}{16\,a^2c^4} = \frac{\cancel{8}\,\cancel{a}\cancel{c^2}}{\underset{3}{\cancel{9}}\,\cancel{d}} \cdot \frac{\overset{5\,d}{\cancel{15}\,\cancel{d^2}e}}{\underset{2\,a\,c^2}{\cancel{16}\,a^2\cancel{c^4}}} = \frac{5\,de}{6\,ac^2}$.

To find the product of two or more fractions or mixed expressions we have the

Rule. *If there are integral or mixed expressions, reduce them to fractional form.*

Separate each numerator and each denominator into its prime factors.

Cancel the factors common to any numerator and any denominator.

Write the product of the factors remaining in the numerator over the product of the factors remaining in the denominator.

RE

ORAL EXERCISES

Simplify:

1. $\frac{1}{2}\cdot\frac{1}{3}$.
2. $\frac{1}{5}\cdot\frac{1}{4}$.
3. $\frac{2}{5}\cdot\frac{1}{3}$.
4. $\frac{3}{7}\cdot\frac{7}{2}$.
5. $\frac{5}{6}\cdot\frac{12}{7}$.
6. $\frac{8}{11}\cdot\frac{22}{10}$.
7. $(\frac{9}{10})^2$.
8. $\frac{a}{d}\cdot\frac{d}{c}$.
9. $\frac{c}{x}\cdot\frac{x}{c}$.
10. $\frac{m}{n}\cdot\frac{n^2}{m^2}$.
11. $\frac{x^2}{y^2}\cdot\frac{y}{x}$.
12. $\frac{x^2}{y^3}\cdot\frac{y}{x}$.
13. $\left(\frac{c^2}{d}\right)^2$.
14. $\frac{2x}{5}\cdot\frac{10}{3x}$.
15. $\left(\frac{5s^2}{t}\right)^2$.
16. $\left(\frac{2a}{3b^2}\right)^3$.
17. $\left(\frac{x+y}{2x^2}\right)^2$.
18. $\frac{m-n}{mn}\cdot\frac{m}{m+n}$.
19. $\frac{2x+c}{xc}\cdot\frac{xc^2}{x-2c}$.

EXERCISES

Multiply:

1. $\frac{12}{15}\cdot\frac{10}{20}$.
2. $\frac{4}{21}\cdot\frac{12}{10}$.
3. $\frac{4}{9}\cdot\frac{15}{22}\cdot 3$.
4. $1\frac{1}{6}\cdot\frac{8}{21}$.
5. $-\frac{14}{11}\cdot\frac{33}{42}$.
6. $2\frac{4}{7}\cdot -1\frac{17}{18}$
7. $2\frac{1}{4}\cdot\frac{1}{21}\cdot 1\frac{7}{8}$.
8. $-1\frac{1}{11}\cdot 1\frac{11}{21}$.
9. $\frac{14}{24}\cdot 16\cdot\frac{32}{70}$.
10. $\frac{22}{7}\cdot(6)^2\cdot\frac{1}{144}$.
11. $\frac{4}{3}\cdot\frac{22}{7}\cdot(2\frac{1}{2})^3$.
12. $\frac{5a^2b}{3c^5}\cdot\frac{6c^2}{10ab^3}$.
13. $\frac{10xy^2}{18z^2}\cdot\frac{24z^3}{15x^4}$.
14. $\frac{15ax^3}{4x}\cdot\frac{12a^5}{40x^2}\cdot\frac{16x^2}{9a^5}$.
15. $\frac{8cd^2}{3x}\cdot\frac{18}{40c^3d^5}\cdot 5x^2d$.
16. $\left(\frac{5x}{2ac}\right)^2\cdot\frac{6a^2}{10x^3}\cdot\frac{5c^5}{3a^4}$.
17. $21x^2y\cdot\frac{9ay^2}{14x^8}\left(\frac{2x}{3y^2}\right)^3$.
18. $\left(\frac{a}{5c^2}\right)^2\cdot\left(\frac{-c}{2d}\right)^3\cdot\frac{15d}{ac}$.
19. $\frac{10m}{n^7}\cdot\left(\frac{4n^2}{15m}\right)^2\cdot\left(\frac{5m^3}{6m^2n}\right)^2$.
20. $\frac{(2x)^2}{3y}\cdot\frac{(3y)^2}{2x^2}$.
21. $\frac{(6a)^2}{(10b^2)^3}\cdot\frac{125b^4}{2(a^3)^2}$.
22. $\left(\frac{8m^2}{10mn^2}\right)^2\cdot\left(\frac{15n^3}{4m}\right)^3$.
23. $\frac{a^2-9}{a^2-2a-15}\cdot\frac{a-4}{2a^3-8a^2}\cdot\frac{2a^3-10a^2}{a^2-6a+9}$.

Solution. $$\frac{a^2-9}{a^2-2a-15}\cdot\frac{a-4}{2a^3-8a^2}\cdot\frac{2a^3-10a^2}{a^2-6a+9}=$$

$$\frac{\cancel{(a-3)}\cancel{(a+3)}}{\cancel{(a+3)}\cancel{(a-5)}}\cdot\frac{\cancel{a-4}}{\cancel{2a^2}\cancel{(a-4)}}\cdot\frac{\cancel{2a^2}\cancel{(a-5)}}{\cancel{(a-3)}(a-3)}=$$

$$\frac{1}{a-3}.$$

Check. Let $a = 2$.

$$\frac{a^2-9}{a^2-2a-15}\cdot\frac{a-4}{2a^3-8a^2}\cdot\frac{2a^3-10a^2}{a^2-6a+9}=\frac{1}{a-3}.$$

$$\frac{-5}{-15}\cdot\frac{-2}{-16}\cdot\frac{-24}{1}=\frac{1}{-1}.$$

$$\frac{1}{3}\cdot\frac{1}{8}\cdot-24=-1.$$

$$-1=-1.$$

24. $\dfrac{2x^2+6}{5y^3}\cdot\dfrac{10y^4}{3x^2+9}.$

25. $\dfrac{c^2-4}{x-y}\cdot\dfrac{3x-3y}{c^2+6c+8}.$

26. $\dfrac{x^2-2x+4}{x^2-9}\cdot\dfrac{x^2-x-6}{x^3+8}.$

27. $\dfrac{x^2-3x+9}{2x-6}\cdot\dfrac{x^2-9}{x^3+27}.$

28. $\dfrac{x^2-7x+10}{x^3-125}\cdot\dfrac{2x^3+10x^2+50x}{x^2-4}.$

29. $\dfrac{x^4-16}{(x^2-4)^2}\cdot\dfrac{x^2-4}{4+x^2}.$

30. $\dfrac{c^2-9}{a^2-4x^2}\cdot\dfrac{a^2+4ax+4x^2}{ac+2cx-3a-6x}.$

31. $\dfrac{2a^2-5ac-3c^2}{c^2-9a^2}\cdot\dfrac{3c+9a}{10a^2+5ac}.$

32. $\dfrac{2x^2+10x+50}{3x^3-12x}\cdot\dfrac{x^2-3x-10}{4x^3-500}\cdot\dfrac{2-x}{2}.$

33. $\dfrac{(x^2-4)(x+3)}{2x^2+12x+18}\cdot\dfrac{x^2-x-6}{(x^2-4)^2}\cdot\dfrac{(x-2)}{x+3}.$

34. $\dfrac{4e^2-12ex+9x^2}{(e^2+ex+x^2)(3x-2e)}\cdot(e^3-x^3)\cdot\dfrac{x-e}{2e^2-5ex+3x^2}.$

35. $\left(4+\frac{3}{a^2-1}\right)\left(\frac{3a}{2a-1}-1\right)$.

HINT. $\left(4+\frac{3}{a^2-1}\right)\left(\frac{3a}{2a-1}-1\right)=\frac{4a^2-1}{a^2-1}\cdot\frac{a+1}{2a-1}$ etc.

The student should note particularly the difference between the procedure in multiplying two expressions like the above and that followed in adding them or in subtracting one from the other. (See Exercise 22, p. 166.)

36. $\left(\frac{x^2}{y^2}-4\right)\left(\frac{2x+4y}{x^3-8y^3}\right)\left(\frac{x^2+2xy+4y^2}{x^2+4xy+4y^2}\right)$.

37. $\left(3x-5-\frac{168}{9x+15}\right)\left(3-\frac{4}{3+x}\right)$.

38. $\left(3x-4+\frac{21}{4+x}\right)\left(\frac{1}{x+1}\right)^2\left(x+\frac{x-3}{3x+5}+3\right)$.

39. $\left(2x-\frac{1}{2x}\right)\left(6x+\frac{3}{2x+1}\right)\left(\frac{1}{8}\right)\frac{1}{8x^3-1}$.

40. $\left(\frac{4}{x^2}-\frac{5}{x}+1\right)\left(\frac{5x^4+5x^3}{x^2-11x+28}\right)\left(1-\frac{7x-1}{x^2-1}\right)$.

41. $\left(2b+\frac{a^2}{2b}-2a\right)\left(a+\frac{12b^2}{a-2b}+4b\right)\cdot\frac{2b^2}{a^3-8b^3}$.

80. Division of fractions. In arithmetic $\frac{3}{4}\div\frac{5}{7}=\frac{3}{4}\cdot\frac{7}{5}=\frac{21}{20}$; and $\frac{3}{4}\div 11=\frac{3}{4}\cdot\frac{1}{11}=\frac{3}{44}$. Also $\frac{3}{4}\div 1\frac{4}{7}=\frac{3}{4}\div\frac{11}{7}=\frac{3}{4}\cdot\frac{7}{11}=\frac{21}{44}$.

Similarly, $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$; and $\frac{a}{b}\div n=\frac{a}{b}\cdot\frac{1}{n}=\frac{a}{bn}$.

Also $\frac{a}{b}\div\left(c+\frac{n}{d}\right)=\frac{a}{b}\div\left(\frac{cd+n}{d}\right)=\frac{a}{b}\cdot\frac{d}{cd+n}=\frac{ad}{bcd+bn}$.

For division of fractions we have the

Rule. *Reduce all integral or mixed expressions to fractional form.*

Then invert the divisor, or divisors, and proceed as in multiplication of fractions.

EXERCISES

Perform the indicated operations:

1. $\frac{2}{3} \div \frac{5}{3}$.
2. $\frac{4}{5} \div \frac{9}{5}$.
3. $\frac{10}{3} \div \frac{5}{2}$.
4. $\frac{12}{15} \div (-\frac{4}{5})$.
5. $\frac{15}{21} \div -\frac{3}{7}$.
6. $\frac{18}{20} \div (-\frac{45}{10})$.
7. $\dfrac{a}{2x} \div \dfrac{c}{x}$.
8. $\dfrac{2a}{b} \div \left(\dfrac{-3c}{2b}\right)$.
9. $\dfrac{x^2}{y} \div \dfrac{2x}{y^2}$.
10. $\dfrac{2m}{n} \div \dfrac{-m}{3n}$.
11. $\dfrac{10x^2}{14y^2} \div \dfrac{15x^3}{21y^4}$.
12. $\dfrac{12ab^2}{5c} \div \dfrac{18a^2b}{15c^3}$.
13. $\dfrac{4a^2x}{51c} \div \dfrac{2ax}{17c^3}$.
14. $\left(\dfrac{3ab}{5c}\right)^2 \div \dfrac{18a^3b}{100c^3}$.
15. $\dfrac{a}{b} \div \dfrac{c}{b} \div \dfrac{a^2}{c^2}$.
16. $\dfrac{2x^6}{5a} \div \dfrac{4x^2}{15} \div \dfrac{(3x^2)^2}{12a^2}$.
17. $\dfrac{12c^3d}{35e^4} \div \dfrac{8c^2}{7e^2d} \cdot \dfrac{10de}{3c^2d^3}$.
18. $\dfrac{18x^3}{(7ab^2)^2} \div 12x^2 \cdot \dfrac{(14ab^4)^2}{6x^2}$.
19. $\left(\dfrac{3}{5x^2}\right)^2 \div \dfrac{256}{(10x^2)^2} \div \dfrac{5x^3}{(8x^4)^2}$.
20. $\dfrac{2a}{3b} \div \dfrac{(2a)^3}{12b^3} \div \dfrac{(7a^4)^2}{15b^5} \cdot \left(\dfrac{7}{5}\right)^2$.

HINT. See Rule, page 13.

21. $\dfrac{10-4a}{12x^2} \div \dfrac{5a-2a^2}{9x}$.
22. $\dfrac{(a-3)^2}{a^2-4a+3} \div \dfrac{a^2-9}{a^2-a}$.
23. $\dfrac{x^2+x-30}{x^2+5x-6} \div \dfrac{x^2-25}{x^2-6x+5}$.
24. $\dfrac{a^3-1}{a^3-a} \div \dfrac{a^4+a^3+a^2}{a^2+3a+2}$.
25. $(2m^2-4m+8) \div \dfrac{m^4+8m}{3m}$.
26. $\left(x+1-\dfrac{3}{4x}\right)\left(3+\dfrac{3}{4x^2-1}\right) \div \dfrac{2x+3}{2x+1}$.
27. $\left(\dfrac{-c}{2d^2}\right)^2 \div \left(\dfrac{-d}{c^3}\right)^2 \cdot \left(\dfrac{-d^3}{c^2}\right)^3 \div \left(\dfrac{d}{-2c}\right)^2$.

28. $\left(\frac{x^2}{y}-\frac{y^2}{x}\right)\div\left(\frac{x^2+xy+y^2}{2x^2-2xy}\right)\cdot\frac{1}{2(x-y)^2}$.

29. $\frac{x^2-5xy+6y^2}{9x^2-6xy+y^2}\cdot\left(3x+5y+\frac{10y^2}{x-2y}\right)\div\frac{x^2-9y^2}{3x^2+8xy-3y^2}$.

30. $\frac{(2m)^2}{m^6+64}\cdot\left(m^2-4+\frac{16}{m^2}\right)\div\frac{1}{2m}\div\frac{4m}{4-m^2}$.

31. $\frac{3x-2}{x+1}\cdot\left(3x+4+\frac{4}{3x}\right)\div\frac{9x^2-4}{9x^2+9x}$.

32. $\left(2x+1+\frac{1}{2x}\right)\div\frac{(8x^3-1)^2}{2x^2+2x}\cdot\left(4x-2+\frac{3}{x+1}\right)$.

33. $\frac{27a^3+8b^3}{ax}\cdot\left(\frac{3a}{2b}\div\frac{2b}{3a}\right)\div\left[\left(3a-2b+\frac{4b^2}{3a}\right)\frac{9ab}{x}\right]$.

81. Complex fractions. A complex fraction is a fractional expression containing one or more fractions either in the numerator, or in the denominator, or in both.

EXAMPLE

Simplify $\dfrac{1+\frac{8}{a}-\frac{9}{a^2}}{a-3+\frac{2}{a}}$.

Solution. Reducing the numerator and denominator to simple fractions,

$$\frac{1+\frac{8}{a}-\frac{9}{a^2}}{a-3+\frac{2}{a}}=\frac{\frac{a^2+8a-9}{a^2}}{\frac{a^2-3a+2}{a}}.$$

Performing the indicated division, the right member becomes

$$\frac{\cancel{(a-1)}(a+9)}{\underset{a}{\cancel{a^2}}}\cdot\frac{\cancel{a}}{(a-2)\cancel{(a-1)}}=\frac{a+9}{a^2-2a}.$$

Check. Letting $a = 3$,

$$\frac{1+\frac{8}{a}-\frac{9}{a^2}}{a-3+\frac{2}{a}} = \frac{a+9}{a^2-2a} \quad \text{becomes} \quad \frac{1+\frac{8}{3}-\frac{9}{9}}{3-3+\frac{2}{3}} = \frac{3+9}{9-6}, \text{ or } 4 = 4.$$

To simplify a complex fraction we have the

Rule. *Reduce both the numerator and the denominator to simple fractions, then perform the indicated division.*

EXERCISES

Simplify:

1. $\dfrac{3+\frac{1}{7}}{1-\frac{1}{7}}$.

2. $\dfrac{\frac{6}{7}-2}{1-\frac{3}{7}}$.

3. $\dfrac{9-\frac{4}{25}}{3-\frac{2}{5}}$.

4. $\dfrac{\left(\frac{3}{2}\right)^2-16}{\frac{3}{2}+4}$.

5. $\dfrac{3^3-\left(\frac{1}{2}\right)^3}{3^2+3\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2}$.

6. $\dfrac{\left(\frac{3}{5}\right)^2+\frac{3}{5}-2}{\frac{3}{5}-1}$.

7. $\dfrac{\left(\frac{2}{3}\right)^3-\left(\frac{3}{2}\right)^3}{\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)^2}$.

8. $\dfrac{\left(\frac{7}{2}\right)^3-\left(\frac{1}{3}\right)^3}{\left(\frac{7}{2}\right)^2+\frac{7}{2}\cdot\frac{1}{3}+\left(\frac{1}{3}\right)^2}$.

9. $\dfrac{\frac{2}{3}}{5}+\dfrac{2}{\frac{3}{5}}$.

10. $\dfrac{c-\frac{1}{c}}{\frac{c-1}{c^2}}$.

11. $\dfrac{2-\frac{c^2}{2}}{\frac{2-c}{4}}$.

12. $\dfrac{x+\frac{x^2}{y}}{1+\frac{x}{y}}$.

13. $\dfrac{\frac{b^2}{5}-5}{1+\frac{b}{5}}$.

14. $\dfrac{\frac{c^2}{d^2}-\frac{d^2}{c^2}}{\frac{1}{c^2}+\frac{1}{d^2}}$.

15. $\dfrac{\frac{a^2}{b}-\frac{b^2}{a}}{a+b+\frac{b^2}{a}}$.

16. $\dfrac{\frac{x}{y}}{z}-\dfrac{x}{\frac{y}{z}}$.

17. $\dfrac{\frac{x}{y}+5+\frac{6y}{x}}{3+\frac{2x}{y}-\frac{9y}{x}}$.

18. $\dfrac{\dfrac{8}{a}-a^2}{1+\dfrac{2}{a}+\dfrac{a}{2}}.$

19. $\dfrac{1-\dfrac{y}{x}}{\dfrac{x^2+y^2}{x}-2y}.$

20. $\dfrac{\dfrac{(x+3y)^2}{6xy}-2}{\left(\dfrac{1}{y}-\dfrac{3}{x}\right)^2}.$

21. $\dfrac{\left(\dfrac{3a-b}{2a^3}\right)^2}{\dfrac{(b-3a)^2}{4a^2}}.$

22. $\dfrac{\dfrac{x-2y}{2y}+\dfrac{x+2y}{x}}{\dfrac{x-2y}{x}-\dfrac{x+2y}{2y}}.$

23. $\dfrac{\dfrac{1}{a}-\dfrac{b}{a+b}}{\dfrac{1}{a}+\dfrac{1}{b}}.$

24. $\dfrac{\dfrac{1}{x^2}-\dfrac{1}{xy}+\dfrac{1}{y^2}}{\dfrac{1}{x^3}+\dfrac{1}{y^3}}.$

25. $\dfrac{\dfrac{1}{5b^2}+\dfrac{1}{2ab}+\dfrac{1}{5a^2}}{\dfrac{1}{2a}+\dfrac{1}{b}}.$

26. $\dfrac{\dfrac{x}{x-1}-\dfrac{2x}{x-2}}{\dfrac{2x}{x-2}-\dfrac{3x}{x-3}}.$

27. $\dfrac{\dfrac{2(b+d)-3(2a-c)}{2ab-bc+2ad-cd}}{\dfrac{2}{2a-c}-\dfrac{3}{b+d}}.$

28. $\dfrac{\left(\dfrac{5x+4y}{4x}\right)^2-\dfrac{5y}{x}}{\dfrac{(5x+4y)^2}{4x}-20y}.$

29. $\dfrac{2+\dfrac{12-6a}{2a^2-a-6}}{2a+\dfrac{21}{2a+3}-7}.$

CHAPTER XVI

EQUATIONS CONTAINING FRACTIONS

82. Equations containing fractions with monomial denominators. If fractions are involved in one or both members of an equation, it is necessary to find a number or a literal expression by which one may multiply both members in order to get rid of the fractions. (Compare Problem 24, p. 145.) This process involves the application of Axiom III, page 40, which is the only principle employed in this chapter that has not been used repeatedly in the earlier work with equations.

Especial care is required to avoid error when a fraction which has two or more terms in its numerator is preceded by a minus sign.

ORAL EXERCISES

Solve for x, stating in each case the operations employed:

1. $\frac{x}{2}=1.$	**5.** $\frac{3x}{2}=4.$	**9.** $\frac{x}{2}-5=0.$	**13.** $\frac{2x}{3}-8=0.$
2. $\frac{x}{3}=2.$	**6.** $\frac{2x}{3}=3.$	**10.** $\frac{x}{4}-7=0.$	**14.** $\frac{3x}{4}+3=0.$
3. $\frac{x}{4}=1.$	**7.** $\frac{2x}{5}=10.$	**11.** $\frac{x}{3}+2=0.$	**15.** $\frac{5x}{3}-10=0.$
4. $\frac{x}{2}=6.$	**8.** $\frac{3x}{4}=9.$	**12.** $\frac{x}{5}+1=0.$	**16.** $\frac{x+1}{2}=2.$

17. $\frac{x-4}{5}=3.$

18. $\frac{x+6}{2}=8.$

19. $\frac{3x+1}{2}=2.$

20. $\frac{2x-1}{3}=5.$

21. $\frac{2x-5}{5}=3.$

22. $1=\frac{2}{x}.$

23. $3=\frac{6}{x}.$

24. $\frac{4}{x}=2.$

25. $\frac{10}{2x}=5.$

26. $\frac{6}{4x}=3.$

27. $\frac{5}{2x}=10.$

28. $\frac{4}{3x}=2.$

29. Using the least multiples possible, free Exercises 1–10, page 177, of fractions.

EXAMPLES

1. Solve the equation $\frac{x}{3}+\frac{x}{5}=16.$

Solution. Multiplying both members by 15, the L.C.M. of the denominators, and canceling, we obtain

$$5x+3x=240,$$

or

$$8x=240.$$

$$x=30.$$

Check. Substituting 30 for x in the given equation,

$$\tfrac{30}{3}+\tfrac{30}{5}=16,$$

or

$$10+6=16.$$

2. Solve the equation $\frac{3}{4}(x+1)-\frac{5x-7}{6}=\frac{7}{3}.$

Solution. Removing the parenthesis,

$$\frac{3x}{4}+\frac{3}{4}-\frac{5x-7}{6}=\frac{7}{3}.$$

Multiplying both members by 12, the L.C.M. of the denominators, and canceling, with particular attention to the signs in the third fraction, we obtain

$$9x+9-10x+14=28.$$

Solving,

$$x=-5.$$

Check. Substituting -5 for x in the given equation,

$$\frac{3}{4}(-5+1)-\frac{-25-7}{6}=\frac{7}{3}.$$

$$-3+\frac{16}{3}=\frac{7}{3}.$$

$$\frac{-9+16}{3}=\frac{7}{3}.$$

$$\frac{7}{3}=\frac{7}{3}.$$

For solving equations containing fractions with monomial denominators, we have the

Rule. *Free the equation of any parentheses it may contain.*

Find the L.C.M. of the denominators of the fractions and multiply each term of the equation by it, using cancellation wherever possible.

Transpose and solve as usual.

EXERCISES

Solve for x, and check results as directed by the teacher:

1. $\frac{x}{2}+\frac{x}{3}=10.$
2. $\frac{x}{4}+\frac{x}{8}=3.$
3. $\frac{x}{3}-\frac{x}{8}=5.$
4. $x+\frac{x}{5}=6.$
5. $\frac{x}{5}+\frac{x}{3}-\frac{x}{15}=14.$
6. $\frac{x+5}{4}-\frac{2x+4}{9}=1.$
7. $\frac{x-2}{2}+\frac{3x+2}{2}=6.$
8. $\frac{x+3}{4}+\frac{4x-5}{5}=5.$
9. $\frac{x+1}{2}+\frac{x-2}{3}-\frac{x+3}{4}=2.$
10. $\frac{2}{3}x+\frac{5}{4}x=\frac{23}{24}.$
11. $\frac{4}{3}x+\frac{2}{5}x=26.$
12. $\frac{3}{2}x+\frac{4}{3}x=\frac{17}{9}.$
13. $\frac{1}{x}+\frac{2}{x}=3.$
14. $\frac{3}{x}-\frac{2}{x}=6.$

15. $\frac{5}{x}+\frac{4}{x}=9.$

16. $\frac{2}{x}+\frac{16}{15}=\frac{19}{3x}.$

17. $\frac{2x+3}{5}-\frac{1}{3}(x-3)=2.$

18. $\frac{5-x}{2}+\frac{2x+3}{3}=4.$

19. $\frac{2x+13}{3}-\frac{6-x}{4}=1.$

20. $\frac{3}{4}(x-1)+\frac{5x-7}{4}=\frac{3}{2}.$

21. $\frac{5x+12}{6}-\frac{4}{11}(2x+7)+\frac{1}{3}=0.$

22. $\frac{x-7}{6}+\frac{1}{2}\left(2-\frac{x}{2}\right)=\frac{3x-22}{6}.$

23. $\frac{x}{a}+x=a+1.$

24. $\frac{x}{a}-\frac{x}{3a}=6.$

25. $\frac{x}{a}-\frac{2x}{5a}=\frac{3}{5b}.$

26. $6-\frac{x}{c}=\frac{1}{3}(c-3x)+\frac{17c}{3}.$

27. $n\left(\frac{2}{3}-\frac{n}{5}\right)-\frac{x}{3}=\frac{n}{5}(5n-3x).$

28. $2x+2a+\frac{3x+4}{a}=\frac{2(2a^2-1)}{a}-1.$

29. $\frac{x+a}{a}-\frac{x+b}{b}=a^2(b-a).$

30. $\frac{2a-3x}{6a}+\frac{5a-2x}{5a}+\frac{41}{30}=0.$

31. $\frac{6x}{a}-\frac{1}{2}\left(\frac{2a}{3}-12x\right)+\frac{19a}{3}=-6.$

32. $(x+5)(x-6)=x\left(x-\frac{5}{2}\right).$

33. $(x+7)(x+18)=x\left(x-\frac{1}{5}\right).$

34. $(4x+3)(6+x)=4x\left(x-\frac{1}{6}\right)+45\frac{2}{3}.$

PROBLEMS

1. One third of a certain number, plus $\frac{1}{12}$ of that number, equals 15. Find the number.

HINT. $\frac{n}{3}+\frac{n}{12}=15.$

2. The difference between $\frac{1}{4}$ of a certain number and $\frac{1}{9}$ of that number is 10. Find the number.

3. The sum of two numbers is 38. One tenth of the greater number equals $\frac{1}{9}$ of the less. Find the numbers.

4. The width of a rectangle is $\frac{2}{3}$ of its length. The perimeter is 200 meters: Find the area of the rectangle.

5. What number must be added to the numerator of the fraction $\frac{4}{7}$ so that the resulting fraction will be $\frac{1}{5}$ of the number in question?

6. One half of a certain integer is $\frac{1}{5}$ the sum of the next two consecutive integers. Find the first integer.

7. A certain even integer divided by 12 is equal to $\frac{1}{26}$ of the sum of the next two consecutive even integers. Find the first integer.

8. What number added to both terms of the fraction $\frac{5}{7}$ gives a fraction whose value is $\frac{5}{6}$?

9. Separate 24 into two parts such that $\frac{1}{4}$ of their difference is 4.

10. One fourth the difference of three times a certain number and 4 equals $\frac{1}{7}$ the difference of five times the number and 4. Find the number.

11. Separate 108 into two parts such that their quotient is $\frac{2}{7}$.

12. There are two numbers whose sum is 24. If their difference be divided by their sum, the quotient will be $\frac{1}{60}$ of the greater number. Find the numbers.

13. A's age is $\frac{5}{2}$ B's age. In 10 years A's age will be twice B's age. Find their ages now.

14. At the time of marriage a certain woman's age was $\frac{4}{5}$ that of her husband. Twenty years later her age was $\frac{8}{9}$ of his. Find their ages at the time of marriage.

15. A is 12 years older than B. Eight years ago B was $\frac{5}{8}$ as old as A. Find their ages now.

16. Jupiter has 5 more moons than Uranus, and Saturn two more than twice as many as Uranus; Mars has 7 fewer than Jupiter, and Neptune half as many as Mars. These planets together have 26 moons. How many has each?

17. A triangle has the same area as a trapezoid. The altitude of the triangle is 24 feet and its base is 12 feet. The altitude of the trapezoid is a third that of the triangle, and one of its bases equals the base of the triangle. Find the other base of the trapezoid.

In solving Problems 18 and 19, the student should make a table similar to those shown on pages 100 and 101.

18. A marksman hears the bullet strike the target 3 seconds after the report of his rifle. If the average velocity of the bullet is 1925 feet per second and the velocity of sound is 1100 feet per second, find the distance to the target and the length of time the bullet was in the air.

19. A gunner using one of the best modern rifles would hear the projectile strike a target 2640 yards distant $9\frac{2}{5}$ seconds after the report of the gun, provided the projectile maintained throughout its flight the same velocity it had on leaving the gun. Find this velocity if sound travels 1100 feet per second.

83. Equations containing fractions with polynomial denominators. Although no new principle is involved in the exercises of this section, they serve to review many of the most important processes of algebra.

ORAL EXERCISES

Clear the following equations of fractions, stating in each case the operation employed. Do not solve the equations.

1. $\dfrac{x}{x+1}=3.$

Solution. Multiplying each member of the equation by $x+1$ we obtain the equation $x=3x+3$.

2. $\dfrac{x}{x-1}=4.$
3. $\dfrac{2x}{1-x}=\dfrac{5}{2}.$
4. $\dfrac{3x}{x^2+4}=1.$
5. $\dfrac{x-1}{x-2}=3.$
6. $\dfrac{x+1}{x-5}=6.$
7. $\dfrac{x+1}{x-1}=\dfrac{4}{3}.$
8. $\dfrac{x+7}{x-3}=\dfrac{5x}{3}.$
9. $\dfrac{x-3}{x+2}=\dfrac{3}{4x}.$
10. $\dfrac{x+1}{x+2}=\dfrac{x+2}{x-1}.$
11. $\dfrac{x+2}{x+3}=\dfrac{x-2}{x-1}.$
12. $\dfrac{x-4}{x+3}=\dfrac{x-5}{x+2}.$
13. $\dfrac{1}{x}=x-1.$
14. $\dfrac{1}{x}=\dfrac{2}{x-1}.$
15. $\dfrac{x-1}{x}=\dfrac{x}{4}.$
16. $\dfrac{x}{x-1}=\dfrac{5}{x}.$
17. $\dfrac{x-2}{3x}=\dfrac{1}{x}.$
18. $\dfrac{4x}{1-2x}=\dfrac{2x}{1+x}.$
19. $\dfrac{3-x}{3x-1}=\dfrac{x}{2-x}.$

EXERCISES

Solve the following equations and check:

1. $2+\dfrac{x-1}{x+3}=\dfrac{11}{5}.$

Solution. Multiply both members of the equation by $5(x+3)$, the L.C.M. of the denominators. This may be indicated symbolically as follows:

$$\left[2+\frac{x-1}{x+3}=\frac{11}{5}\right]5(x+3).$$

By means of this form of expression one can readily see what ought to be canceled.

Canceling, $\quad 10(x+3)+5(x-1)=11(x+3),$

or $\quad 10x+30+5x-5=11x+33.$

Transposing and collecting, $\quad 4x=8,$

whence $\quad x=2.$

Check.

$$2 + \frac{2-1}{2+3} = \frac{11}{5}.$$

$$2 + \tfrac{1}{5} = \tfrac{11}{5}.$$

$$\tfrac{11}{5} = \tfrac{11}{5}.$$

2. $\dfrac{x-3}{3+x} = 4.$ 3. $\dfrac{x-5}{x-2} = \dfrac{1}{2}.$ 4. $\dfrac{3x+4}{2x-2} = -2.$

5. $\dfrac{4}{6-5x} = \dfrac{1}{x}.$ 6. $\dfrac{5x}{3} = \dfrac{3-x}{2+x} + x.$

Solution. The L.C.M. of the denominators is $3(2+x)$. Hence from an inspection of

$$\left[\frac{5x}{3} = \frac{3-x}{2+x} + x\right] 3(2+x)$$

we obtain $$5x(2+x) = 3(3-x) + 3x(2+x),$$

or $$10x + 5x^2 = 9 - 3x + 6x + 3x^2.$$

Transposing and collecting,

$$2x^2 + 7x - 9 = 0.$$

Factoring, $$(2x+9)(x-1) = 0,$$

whence $$x = -\tfrac{9}{2}, 1.$$

Check as usual.

7. $\dfrac{x-3}{x} = \dfrac{2x}{x-2}.$

8. $\dfrac{x-1}{x+3} = x.$

9. $\dfrac{x}{5-x} = \dfrac{2x}{4x-5}.$

10. $\dfrac{2}{x+1} = \dfrac{4}{4+x}.$

11. $\dfrac{1}{2-x} = \dfrac{3}{3-x}.$

12. $\dfrac{x-5}{4-5x} = -6x.$

13. $\dfrac{x+7}{3-x} + \dfrac{3x}{2} = x.$

14. $\dfrac{2}{x-6} = \dfrac{x-12}{x+6} + \dfrac{x}{x+6}.$

15. $\dfrac{x+3}{x-4} = \dfrac{x+9}{x-5}.$

16. $\dfrac{4x-1}{x+2} = \dfrac{7-2x}{x+4}.$

17. $\dfrac{4}{2x-3} + \dfrac{3x+2}{3} = x.$

18. $\dfrac{x+6}{3-x} = \dfrac{5+2x}{7x-5}.$

19. $\dfrac{3x}{8} - \dfrac{4}{x-4} = \dfrac{3x-4}{8}.$

20. $\dfrac{2}{4x+1} - \dfrac{1}{5x-2} - \dfrac{1}{3x+4} = 0$

21. $\dfrac{47}{220} - \dfrac{7}{4(x+3)} = \dfrac{3}{5(x+3)}.$

22. $\dfrac{4x}{x+3} - \dfrac{6}{2(x+3)} = \dfrac{x^2+11}{3(x+3)}.$

23. $\dfrac{4-x}{x-5} + \dfrac{7}{5} = \dfrac{3}{5-x}.$

HINT. Multiply the terms of the last fraction by -1.

24. $\dfrac{x}{x-3} + 3 = \dfrac{1}{3-x}.$

25. $\dfrac{3x}{x-2} = \dfrac{x-3}{2-x} - x.$

26. $\dfrac{3x}{x-3} - \dfrac{16}{x^2-9} = \dfrac{x}{x+3} + \dfrac{2x}{9-x^2}.$

HINT. Rewriting the last fraction (see Hint after Exercise 23), and factoring the denominators,

$$\frac{3x}{x-3} - \frac{16}{(x-3)(x+3)} = \frac{x}{x+3} - \frac{2x}{(x-3)(x+3)}.$$

The L.C.M. of the denominators is $(x-3)(x+3)$. Hence from

$$\left[\frac{3x}{x-3} - \frac{16}{(x-3)(x+3)} = \frac{x}{x+3} - \frac{2x}{(x-3)(x+3)}\right](x-3)(x+3)$$

we obtain $\quad 3x(x+3) - 16 = x(x-3) - 2x,$

which may be solved and checked as usual.

In solving equations containing fractions with polynomial denominators the student should write the denominators and their L.C.M. in factored form, as in the preceding work. With this exception the rule on page 177 applies to all equations containing fractions.

27. $\dfrac{1}{3-x} + \dfrac{2}{x+3} = \dfrac{6x}{x^2-9}.$

28. $\dfrac{2x}{x+1} + \dfrac{3}{x-1} + \dfrac{24}{1-x^2} = 0.$

29. $\dfrac{3}{x-4} = \dfrac{5x-12}{x^2-16} - \dfrac{4}{16-x^2}.$

30. $\dfrac{x+2}{x-2}=\dfrac{10-2x^2}{4-x^2}-\dfrac{7}{x^2-4}.$

31. $\dfrac{1+x}{3-x}+\dfrac{x-2}{2-x}=\dfrac{x-1}{x^2-5x+6}.$

32. $\dfrac{x-4}{2x-5}+\dfrac{2x-15}{2x+4}=\dfrac{8x^2-20x-31}{4x^2-2x-20}.$

84. Equations containing decimals. The method of solving an equation containing decimals is illustrated in the following:

EXAMPLE

Solve the equation $.3x+.7=4.2-.05x$.

Solution. Multiplying each member of the equation by 100,

$$30x+70=420-5x.$$

Transposing and combining,

$$35x=350.$$
$$x=10.$$

Check.
$$.3\times 10+.7=4.2-.05\times 10.$$
$$3+.7=4.2-.5.$$
$$3.7=3.7.$$

In equations containing fractions, if decimals occur in any denominator, multiply both numerator and denominator of such a fraction by that power of ten which will reduce the decimals in both the numerator and the denominator to integers. Then clear of decimals and proceed as in the foregoing example. (See Hint after Exercise 20, p. 185.)

EXERCISES

Solve and check the following equations:

1. $.4x=6.$
2. $.3x+.5=.8.$
3. $.3x+4=.25.$
4. $.75-.7x=.26.$
5. $.92+.3x=5.12.$
6. $3.75=2.15-.5x.$

7. $.06\,x - 4.5 = 1.68.$

8. $.03\,x + .16 = .58.$

9. $.04\,x = .1x + 2.4.$

10. $.8\,x - 2.7 = .55\,x.$

11. $.15\,x - .4\,x = 8\,x - 49.5.$

12. $1.7\,x + 3.14 = - 9.66 - 1.5\,x.$

13. $3\,x + 7 - 1.25\,x = 8.845 + .52\,x.$

14. $1.7\,x + .17 - .03\,x = 4.73 + 1.1\,x.$

15. $.12(2\,x + .5) - .2(1.5\,x - 2) = .4.$

16. $6(3\,x - 1.1) - 8.4(.7\,x - 3) = 6\,x + .24.$

17. $\dfrac{.7\,x}{4} - \dfrac{.19}{3} = \dfrac{.1\,x}{6}.$

18. $\dfrac{.3\,x - 6.2}{4} - \dfrac{6.75 - .4\,x}{5} = \dfrac{3.5}{2}.$

19. $\dfrac{.14\,x + 3.2}{6} - \dfrac{x - .75}{5} = \dfrac{2.4\,x - 1.5}{10}.$

20. $\dfrac{6\,x - 2.69}{.4} + \dfrac{1.5\,x}{.24} = 3.9.$

HINT. Multiplying numerator and denominator of each fraction by 100,

$$\frac{600\,x - 269}{40} + \frac{150\,x}{24} = 3.9.$$

Solve and check as usual.

To avoid the possibility of repeating, in the check, a numerical error made in the solution, the check should be performed by finding the value of each fraction separately, without clearing of fractions.

21. $\dfrac{1.8\,x - 2}{1.7} = \dfrac{5.7 - .7\,x}{1.8}.$

22. $\dfrac{3.2\,x}{.5} + \dfrac{4.5\,x}{12.5} = 1.52 + 6\,x.$

23. $\dfrac{.3(3 - .4\,x)}{.16} - \dfrac{.5(.2\,x - 6)}{.8} = 5.$

24. $\dfrac{.3(5 - x)}{6.25} = \dfrac{1.5 - 10\,x}{14} + .56\,x.$

NOTE. The introduction into Europe of the Arabic notation for numbers was one of the important events of the Middle Ages. This notation originated among the Hindus at least as early as A.D. 700. It was adopted by the Arabs, and was introduced by the Moors into Spain during the twelfth and thirteenth centuries. Anyone who has tried to multiply two numbers in the Roman notation, like MDCCVII by MCXVIII, will realize the difficulties that surrounded arithmetical operations before the Arabic system was taught. Before the introduction of this system one of the principal uses for arithmetic was the determination of the day of the month on which Easter came. Roger Bacon in the thirteenth century urged the theologians "to abound in the power of numbering," so that they might carry out these computations. Business computations were made on the abacus, one form of which was a contrivance of wires and sliding balls on which arithmetical operations can be performed with great rapidity.

Though computation in the decimal system was common in Europe from the thirteenth century, the final step in perfecting the notation was not taken until about 1600, when Sir John Napier and others made use of the decimal point in the modern sense. It was not until the beginning of the eighteenth century that it came into general use.

85. Percentage. The methods of algebra may be used to advantage in treating many problems in percentage which are also found in arithmetic, and in solving many others which would be difficult or impossible to solve by arithmetical means alone.

ORAL EXERCISES

1. What is 4% of 60?
2. What is 4% of x?
3. What is 3% of $x + 40$?
4. What is 6% of $4x - a$?
5. What is the interest on \$100 left for one year at 5%?
6. What is the interest on \$100 left for 6 years at 5%?
7. What is the interest on \$100 left for t years at 6%?
8. What is the interest on P dollars left for t years at 6%?

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9. What is the interest on P dollars left for t years at $r\%$?

10. What is the total amount due at the end of t years if P dollars are left at $r\%$?

If two sums of money are x dollars and $2500 - x$ dollars respectively, express as an equation the statement made in each of Exercises 11–14.

11. Four per cent of the first sum equals \$24

12. Five per cent of the first sum plus 6% of the second sum equals \$130.

13. Six per cent of the first sum equals 4% of the second.

14. Five per cent of the first sum is \$44 more than 4% of the second.

When P dollars are left at simple interest for t years, at a yearly rate of $r\%$, the total amount A accumulated is given by the following formula:

$$P + (P \cdot r \cdot t) = P(1 + rt) = A.$$

It should be noted carefully that the value of r is a fraction of which the denominator is 100. Thus, if the rate is 5%, the value of r is $\frac{5}{100}$, or .05.

EXAMPLE

What sum of money placed at simple interest for 2 years at 5% amounts to \$99?

Solution. Principal + interest = 99.

Let $P =$ the principal.

Then $.05P \cdot 2 =$ the interest for two years.

Therefore $P + .10P = 99,$

or $1.10P = 99.$

Whence $P = 90.$

Check. $90 + 9 = 99.$

EXERCISES

1. What sum of money placed at interest for 1 year at 6% amounts to \$371?

2. What sum of money placed at simple interest for 3 years at 4% will amount to \$476?

3. In how many years will \$325, at 6% simple interest, gain \$39?

4. In how many years will \$480, at $6\frac{1}{2}$% simple interest, gain \$156?

5. At what per cent simple interest will \$375 gain \$75 in 4 years?

Solution. $375 \times$ the rate of interest $\times 4 = 75$.

Let $x =$ the rate of interest.

Then $\$375\,x =$ the interest for 1 year,

and $\$1500\,x =$ the interest for 4 years.

Therefore $1500\,x = 75$,

and $x = \frac{1}{20}$, or $\frac{5}{100}$.

Hence the money is lent at 5%.

6. At what per cent simple interest will \$725 gain \$145 in 4 years?

7. At what per cent simple interest will \$250 amount to \$317.50 in 6 years?

8. In how many years will \$300 double itself at 5% simple interest?

9. In how many years will \$500 treble itself at 6% simple interest?

10. A part of \$800 is invested at 3%, and the remainder at 4%. The yearly income from the two investments is \$30. Find each investment.

Solution. One part $\times .03 +$ the other part $\times .04 = 30$.

Let $x =$ the number of dollars invested at 3%.

Then $800 - x =$ the number of dollars invested at 4%.

Therefore, by the conditions of the problem,

$$.03\,x + .04\,(800 - x) = 30.$$

Multiplying each member by 100,

$$3\,x + 4\,(800 - x) = 3000.$$

Solving, $\qquad x = 200,$

and $\qquad 800 - x = 600.$

Hence the 3% investment is \$200, and the 4% investment is \$600.

Check.

$$\begin{array}{r} 200 \\ \underline{.03} \\ 6.00 \end{array} \qquad \begin{array}{r} 600 \\ \underline{.04} \\ 24.00 \end{array} \qquad \$6 + \$24 = \$30.$$

11. A part of \$1500 is invested at 5% and the remainder at 4%. The total annual income from the two investments is \$69. Find the amount of each investment.

12. A sum of money at 6% interest and a second sum at 5% yield a total annual income of \$50. The first sum exceeds the second by \$100. Find each.

13. A 4% investment yields annually just as much as one at 5%. If the sum of the investments is \$3600, find each.

14. A 5% investment yields annually \$10 less than a 6% investment. If the sum of the two investments is \$240, find each.

15. A man invests part of \$4300 at 6% and the remainder at 5%. The investment at 6% yields annually \$1.60 less than the one at 5%. Find the sum invested at 5%.

16. A man invests part of \$5360 at 5% and the remainder at 6%. The yearly income from the 5% investment is \$63.40 more than that from the 6% investment. Find the sum invested at 6%.

17. A part of \$4560 is invested at 4% and the remainder at 6%. The total yearly income is \$202.40. Find the amount invested at 6%.

86. Literal equations. At this point the student should review the solutions on pages 95 and 96.

EXERCISES

Solve for x, and check as directed by the teacher:

1. $2ax - 2a^2 = 6a^2 - 2ax.$
2. $5c^2 - 4cx = c(5x - 4c).$
3. $3a^2x - 4b^2 = b^2 - 2a^2x.$
4. $2(x + 3) - 4a = 6.$
5. $3(x - 1) - 9a = 6.$
6. $3(x - 2) - 4h = 2(x - 2).$
7. $3(2x - a) = 2(x - a).$
8. $4(x - b) = 2(x + b).$
9. $4a(6x - 3b) = 3b(8 - 4a).$
10. $ax + bx = a^2 + ab.$
11. $cx + b^2 = bx + bc.$
12. $ax + b^2 = a^2 - bx.$
13. $6ab + 15cx - 10bc - 9ax = 0.$
14. $12 - 15a + 16x = 20ax.$
15. $ax + bx + cx = ak + bk + ck.$
16. $2ax + 2a + cx + c + x + 1 = 0.$
17. $6ab + kx + 4a^2 = 2ax + 3bk + 2ak.$
18. $5ax - 5a^2 - 10ab = 3ac + 6bc - 3cx.$
19. $\dfrac{x}{2b} = a.$
20. $\dfrac{3ab}{x} = b.$
21. $\dfrac{c}{x} + \dfrac{3c}{2x} = \dfrac{5}{4}.$
22. $\dfrac{8a}{3x} + \dfrac{8a}{x} = \dfrac{3}{2} + \dfrac{5a}{3x}.$
23. $\dfrac{x}{a} + \dfrac{x}{b} = a + b.$
24. $\dfrac{d^2}{x} - d = \dfrac{a^2}{x} + a.$
25. $\dfrac{hx}{2k} - 4k^2 = \dfrac{2kx}{h} - h^2.$
26. $\dfrac{2x}{a} + \dfrac{a - 4x}{3} - 3a = -4.$
27. $\dfrac{x}{a} + \dfrac{x}{c} + ac = bc + ab + \dfrac{x}{b}.$
28. $\dfrac{3a - 4x}{5a} + \dfrac{3a - 2x}{4a} = \dfrac{1}{20}.$
29. $\dfrac{bx}{2} - \dfrac{3a}{5}\left(x - \dfrac{2ab}{3}\right) = ab\left(\dfrac{b}{2} - \dfrac{a}{5}\right).$
30. $\dfrac{x - a}{x - b} = \dfrac{b}{a}.$

31. $\frac{b}{b(b-x)}+\frac{3}{d(b-x)}+\frac{d+3}{2\,db}=0.$

32. $\frac{c}{k(x+c)}+\frac{k}{c(x-k)}=\frac{c^2-kc+2\,k^2}{2\,kc(x-k)}.$

33. $\frac{1}{ab}+1-\frac{ab}{x}=\frac{1}{x}.$

87. Meaning of primes and subscripts. Different but related values are often represented by the same letter, with smaller characters written at the right and above or below the letter used; as, y', y'', x_0, $4\,x_3$, t_m^2, t_w. These are read *y prime, y second, x sub zero, 4 x sub three, the square of t sub m, and t sub w* respectively. Primes and subscripts, unlike exponents, possess no numerical significance, and the student should carefully note that x_0 and x_3 are as different numerically as a and b.

The notation just explained is very convenient in physics, where L_1 and L_2 may denote different but related lengths; W_1 and W_2 may represent two different weights; and t_0, t_1, and t_2 may mean three unequal but related intervals of time.

Primes are cumbersome and easily confused with exponents; hence subscripts are preferable.

The following equations are taken from algebra, geometry, and physics, where it is often necessary to express one of the quantities (weight, time, distance, etc.) in terms of the others.

EXERCISES

1. Solve for R; $K=2\,\pi RH$.
(Formula for curved surface of cylinder.)

2. Solve for a; $A=\frac{ab}{2}$.
(Formula for area of triangle.)

3. Solve for R; $C=2\,\pi R$.
(Formula for circumference of circle. $\pi=$ about 22/7.)

4. Solve for r and t; $d = rt$.

(Formula for uniform motion.)

5. Solve for a and A; $\frac{a}{A} = \frac{D}{360}$.

(Formula relating to the measurement of angles.)

6. Solve for C; $\frac{D}{360} = \frac{l}{C}$.

7. Solve for r; $C = \frac{E}{R + r}$.

(Ohm's law for a simple electrical circuit.)

8. Solve for r and n; $C = \frac{E}{R + nr}$.

9. Solve for r and n; $C = \frac{n \cdot e}{R + nr}$.

10. Solve for F; $C = \frac{5}{9}(F - 32)$.

(Formula for converting thermometer readings from one scale (Fahrenheit) to another (Centigrade).)

11. Solve for W_2; $\frac{W_1}{W_2} = \frac{L_2}{L_1}$.

12. Solve for r and t; $A = P(1 + rt)$.

13. Solve for P_2; $\frac{V_1}{V_2} = \frac{P_2}{P_1}$.

(Formula relating to volume and pressure of a gas.)

14. Solve for n and l; $s = \frac{n(a + l)}{2}$.

15. Solve for a, l, and r; $s = \frac{rl - a}{r - 1}$.

16. Solve for a; $\frac{D}{180} = \frac{a}{\pi}$.

17. Solve for t_1; $V_1 = V_0(1 + .00365\, t_1)$.

18. Solve for b_2; $A = \frac{(b_1 + b_2)a}{2}$.

88. The lever. The figure given below is a diagram of a machine called a *lever.* AC is a stiff bar resting on a single support at B. This support is called the *fulcrum* and AB and BC are spoken of as *arms* of the lever.

Those who have played with a teeter board have had some experience with a lever, and they have found that, in order to balance, the heavier of two persons must sit nearer the fulcrum than the lighter one does.

In general, if the lengths of the arms of a lever are l_1 and l_2 and the corresponding weights are W_1 and W_2, a balance results when

$$l_1W_1 = l_2W_2.$$

Thus, if $AB = 3$ feet and $BC = 4$ feet, a boy at A who weighs 100 pounds will balance a boy at C who weighs 75 pounds; for $3 \cdot 100 = 4 \cdot 75$.

PROBLEMS

1. A, who is 4 feet from the fulcrum, balances B, who is 6 feet from it. A weighs 96 pounds. Find the weight of B.

2. A, who weighs 100 pounds, balances B, who weighs 120 pounds. B is 80 inches from the fulcrum. How far from it is A?

3. A, who weighs 125 pounds, balances B, who weighs 100 pounds. The distance between them is 9 feet. How far is each from the fulcrum?

4. A and B together weigh 210 pounds. They balance when A is 3 feet 9 inches from the fulcrum and B is 5 feet from it. Find the weight of each.

5. A's weight of 110 pounds is $\frac{2}{3}$ as much as B's. They are 24 feet apart. How far from B is the fulcrum if they balance?

REVIEW PROBLEMS

1. Separate 240 into two parts such that their quotient is 7.

2. Separate 68 into two parts such that $\frac{2}{3}$ of the greater shall equal $\frac{3}{4}$ of the less.

3. Separate $\frac{4}{3}$ into two parts such that $\frac{1}{3}$ of one part shall equal $\frac{1}{5}$ of the other.

4. Find two numbers whose sum is 98 and such that the greater divided by the less gives a partial quotient of 3 and a remainder of 6.

HINTS. $$\frac{\text{Dividend}}{\text{Divisor}} = \text{Partial Quotient} + \frac{\text{Remainder}}{\text{Divisor}}.$$

Let $x =$ the less number.

Then $98 - x =$ the greater number.

That is, $$\frac{98 - x}{x} = 3 + \frac{6}{x}.$$

Solve and check as usual.

5. Separate 120 into two parts such that one divided by the other gives a partial quotient of 5 and a remainder of 12.

6. The sum of two numbers is 1797. The greater divided by the less gives a partial quotient of 70 and a remainder of 22. Find the numbers.

7. Separate $\frac{11}{6}$ into two parts such that their product is greater by $\frac{1}{3}$ than the square of the smaller part.

8. The sum of two numbers is 16. Four times the greater number exceeds 50 by half as much as 28 exceeds the less. Find the numbers.

9. A boy's age now is $\frac{3}{5}$ of what it will be 10 years hence. How old is he now?

10. One sixth of a certain man's age 12 years ago equals $\frac{1}{8}$ of his age 8 years hence. What is his age now?

11. A collection of nickels and quarters contains 60 coins. Their total value is \$11. How many are there of each?

12. Twenty-three coins, dimes and quarters, have the value \$3.05. How many are there of each?

13. The square of half a certain even number is 14 less than $\frac{1}{4}$ the product of the next two consecutive even numbers. Find the numbers.

14. A rectangle is 4 times as long as it is wide. If it were 4 feet shorter and $1\frac{1}{2}$ feet wider, its area would be 11 square feet more. Find its length and breadth.

15. A rectangle is $\frac{2}{3}$ as broad as it is long. If its length were doubled and its breadth diminished by 14, its area would be diminished by the square of half its breadth. What are its dimensions?

16. It costs as much to sod a square piece of ground at 15 cents per square yard as to fence it at 20 cents per foot. Find the side of the square.

17. A rectangular court is twice as long as it is wide. It costs half as much to fence it at 50 cents per yard as to seed it at 15 cents per square yard. Find its dimensions.

18. A rectangular picture $2\frac{1}{2}$ times as long as wide is surrounded by a frame 2 inches wide. The area of the frame is 128 square inches. Find the dimensions of the picture.

19. A man bought apples at 16 cents per dozen. He sold $\frac{1}{5}$ of them at the rate of 3 for 5 cents, but the rest were not so good, and he had to sell them at the rate of 2 for 3 cents. He made a profit of 48 cents on the entire transaction. How many dozen apples did he buy?

20. A can do a piece of work in 2 days, B in 3 days, and C in 4 days. How long will it take them, working together?

Solution. By the conditions of the problem A does $\frac{1}{2}$ of the work in one day, B does $\frac{1}{3}$ of the work in one day, and C does $\frac{1}{4}$ of the work in one day. Let x represent the number of days required by A, B, and C together to do the work.

Then $\dfrac{1}{x}$ = the fractional part of the work the three together do in one day.

Therefore $$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{1}{x}.$$

Solving, $$x=1\tfrac{1}{13}\!\!\!\!\!\!$$

Check. $$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{1}{\frac{12}{13}}, \text{ or } \frac{13}{12}=\frac{13}{12}.$$

21. A can do a piece of work in 6 days, and B in 9 days. How many days will they require, working together?

22. A can do a piece of work in 6 days, B and C each in 8 days, D in 12 days. How many days will they require, working together?

23. A can do a piece of work in 3 days, B in $4\frac{1}{2}$ days. How many days will they require, working together?

24. A can do a piece of work in 8 days, A and B working together in $4\frac{4}{5}$ days. How long would it take B alone?

25. A can do a piece of work in $5\frac{1}{4}$ days, B in $4\frac{1}{5}$ days, A, B, and C together in $1\frac{3}{4}$ days. How long would it take C alone?

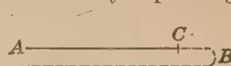
26. A can do a piece of work in 8 days. After he has worked 3 days, B joins him and they finish the work in 3 more days. How long would it have taken B to do the work alone?

Hint. What fractional part does A do in 1 day? in 3 days? in 6 days? What fractional part does B do in 1 day? in 3 days?

27. A can do a piece of work in 4 days, B in 5 days. After A has worked alone for 1 day, B joins him and they finish the job together. How much longer does it take them?

28. Two bicyclists start at the same time to ride from A to B, 80 miles distant. One travels 4 miles an hour more than the other. The faster bicyclist reaches B and at once returns, meeting the slower one at C, 64 miles from A. Find the rate of each.

Solution. The problem states that the two travel at different rates, that they travel different distances, but that the time is the same for each. Hence the equation must be formed by expressing the time t, or d/r, for each, and equating the two expressions for t.



The men together cover twice the distance from A to B, or 160 miles. As the slower one travels 64 miles, the faster travels $160-64$, or 96, miles. If x equals the rate of the slower bicyclist in miles per hour, we have:

	d in miles	r in miles per hour	$\frac{d}{r}=t$ in hours
Slower bicyclist	64	x	$\frac{64}{x}$
Faster bicyclist	$160-64=96$	$x+4$	$\frac{96}{x+4}$

Hence $$\frac{64}{x}=\frac{96}{x+4}.$$

Solving, we obtain $x=8$, the rate of the slower bicyclist in miles per hour, and $x+4=12$, the rate of the faster bicyclist.

Check. $\frac{64}{8}=8$, and $\frac{96}{12}=8$.

29. Two bicyclists, A and B, start at the same time to ride from X to Y, 60 miles distant. A travels 4 miles per hour less than B. The latter reaches Y and at once turns back, meeting A 12 miles from Y. Find the rate of each.

30. A train runs 360 miles. On the return trip it increases its rate by 5 miles an hour and makes the run in an hour less time. Find the rates going and returning.

31. An automobile makes a run of 120 miles. The chauffeur then increases the speed by 5 miles an hour and returns over the same route in 4 hours less time. Find the rates going and returning.

32. A bicyclist traveling 15 miles per hour was overtaken $11\frac{1}{3}$ hours after he started by an automobile which left the same starting point 3 hours and 20 minutes later. What was the rate of the automobile?

33. A man travels at a uniform rate from A to B, 120 miles distant. He travels the first 70 miles without stopping. The remainder of the journey, including a delay of 2 hours, requires the same time as the first part. Find his rate.

Solution. By reading the problem we discover that the distances covered in the first and second portions of the journey are different, that the time of travel is not the same for each, but that the rate throughout is the same. Hence the equation will be formed by finding two expressions for the rate r, or d/t, and setting them equal to each other. If x equals the number of hours required to travel 70 miles, we have:

	d in miles	t in hours	$d/t = r$ in miles per hour
First part of journey	70	x	$\frac{70}{x}$
Second part of journey	50	$x - 2$	$\frac{50}{x-2}$

Hence $$\frac{70}{x} = \frac{50}{x-2}.$$

Solving, we obtain $x = 7$, the time in hours occupied in traveling the first 70 miles, and $70 \div 7 = 10$, the rate in miles per hour.

Check. $70 \div 10 = 7$, and $7 - 5 = 2$.

34. A leaves a certain point and walks at the rate of $3\frac{1}{2}$ miles per hour. Two and a half hours later B leaves the same point

and drives in the opposite direction at the rate of $9\frac{1}{2}$ miles per hour. How much time must elapse after A starts before they will be 25 miles apart?

35. A and B start at the same time from two points 120 miles apart and travel toward each other. A's rate is 2 miles per hour less than B's. The latter, having been delayed $2\frac{1}{2}$ hours on the way, has traveled the same distance as A when they meet. Find the rate of each.

36. A man rows 4 miles per hour in still water. He finds that it requires 5 hours to row upstream a distance which he can row downstream in 3 hours. Find the rate of the current.

HINT. Let $x =$ the rate of the current. Then $4 - x =$ the rate upstream, and $4 + x =$ the rate downstream.

37. A man who can row $4\frac{1}{2}$ miles per hour in still water rows up a stream the rate of whose current is 2 miles per hour. After rowing back he finds that the entire trip took 6 hours. How far upstream did he go?

38. A man who can row 4 miles an hour in still water rows downstream and returns. The rate of the current is $1\frac{1}{3}$ miles per hour, and the time required for the round trip is 15 hours. How many hours did he take to return?

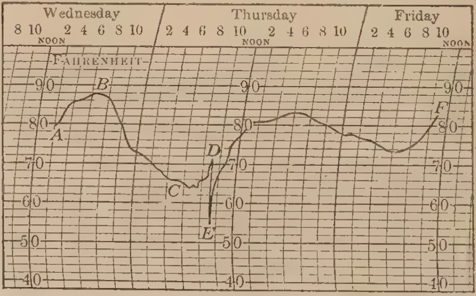
39. French scouts observe a German aëroplane leave its base, fly 24 miles due west over the French lines, and return to its starting point. They are prevented by an encounter with a German detachment from observing the time of either part of the trip, but they are able to observe that the duration of the round trip was 40 minutes. At the time there was an east wind blowing, with a velocity of 15 miles per hour. What was the velocity of the aëroplane?

40. A farmer pays $48 for a flock of sheep. He sells all but two of them for $50, and gains a dollar on each sheep sold. How many sheep had he?

CHAPTER XVII

GRAPHICAL REPRESENTATION

89. Temperature curve. The curve *ABCDEF* is called **a graph.** It was made by a recording thermometer. Such an instrument is provided with an arm carrying a pen, which moves up as the temperature rises, and down as it falls. A clock movement runs a strip of cross-ruled paper under the pen, and thus a continuous line is traced on the paper. The accompanying record extends from 2 P.M. of a certain Wednesday until 10.30 A.M. of the Friday following. The numbers 40, 50, 60, 70, 80, and 90 denote



degrees Fahrenheit. There are 5 spaces from 50° to 60°. Hence one space corresponds to 2 degrees. The numbers 2, 4, 6, 8, and 10 indicate the time of day. Whether this is A.M. or P.M. can be determined by noting the position of these numbers with reference to the heavy curved lines marked NOON. The point *A* on the graph informs us that at 2 P.M. Wednesday the temperature was 80°. The point *B* between 6 P.M. and 7 P.M. Wednesday marks the highest temperature recorded.

The point C tells us that the temperature was about 65° at 6 A.M. Thursday.

The preceding record was made indoors, and the sudden fall from D to E was caused by the opening of a door leading into a cold hallway. The portion of the graph from D to E shows that the temperature of the room fell approximately 18 degrees in about 30 minutes.

ORAL EXERCISES

By reference to the graph (p. 200) answer the following:

1. With what temperature does the record begin? end?

2. What is the highest temperature recorded? the lowest?

3. About what time was the highest temperature recorded? the lowest?

4. How often did the instrument record a temperature of 82°? 76°? 73°? 68°?

5. At what times did it record a temperature of 82°? 76°? 73°? 68°?

6. To what practical use can a graph such as the one here explained be put?

90. Related pairs of numbers. A relation between two sets of numbers not necessarily connected with physical quantities such as temperature and time can also be expressed by a graph, as will be shown later.

The question What two numbers added give four? may be expressed by the equation $x + y = 4$. Here x and y are any two numbers whose sum is 4. It can be seen by inspection that if $x = 2$, $y = 2$; if $x = 1$, $y = 3$; and if $x = 0$, $y = 4$, etc. Or we may assign to x any value, say -2; then the equation becomes $-2 + y = 4$, whence $y = 6$.

In this manner we may obtain an unlimited number of sets of related values for x and y, some of which are given in the following table:

$$x + y = 4$$

	A	B	C	D	E	F	G	H	I
If $x =$	2	1	0	3	4	-1	-2	5	6
then $y =$	2	3	4	1	0	5	6	-1	-2

91. Definitions and assumptions. In constructing the graph of an equation in two variables a number of assumptions must be made. These assumptions and some necessary definitions are now stated. It is agreed:

I. To have two lines at right angles to each other, as $X'OX$, called the **x-axis**, and $Y'OY$, called the **y-axis**, as in the figure on page 203.

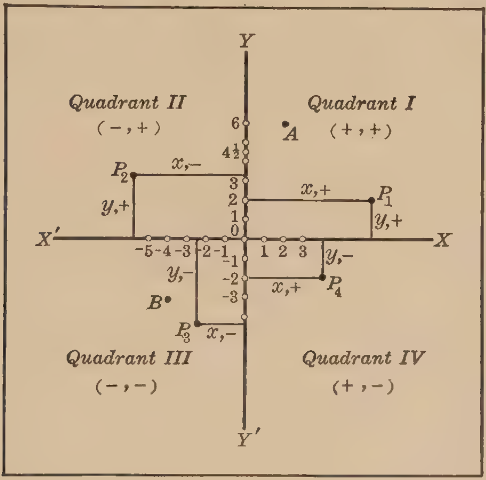
II. To have a line of definite length as a unit of distance. Then the number 2 will correspond to a distance twice the unit in length, the number $4\frac{1}{2}$ to a distance of $4\frac{1}{2}$ times the unit, etc.

III. That the distance (measured parallel to the x-axis) from the y-axis to any point in the surface of the paper be the **x-distance** (or abscissa) of that point, and the distance (measured parallel to the y-axis) from the x-axis to the point be the **y-distance** (or ordinate) of the point.

IV. That the x-distance of a point to the *right* of the y-axis be represented by a *positive* number, and the x-distance of a point to the *left* by a *negative* number; also that the y-distance of a point *above* the x-axis be represented by a *positive* number, and the y-distance of a point *below* the x-axis by a *negative* number. Briefly, *distances measured from the axes to the* ***right*** *or* ***upward*** *are* ***positive***, *to the* ***left*** *or* ***downward*** *are* ***negative***.

V. That every point in the surface of the paper corresponds to a *pair of numbers*, one or both of which may be positive, negative, integral, or fractional.

VI. That of a given pair of numbers the first be the measure of the x-distance, and the second the measure of the y-distance. Thus the point (2, 6), or A in the figure, is the point whose x-distance is 2 and whose y-distance



is 6. Again the point $(-4, -3)$, or B, is the point whose x-distance is -4 and whose y-distance is -3.

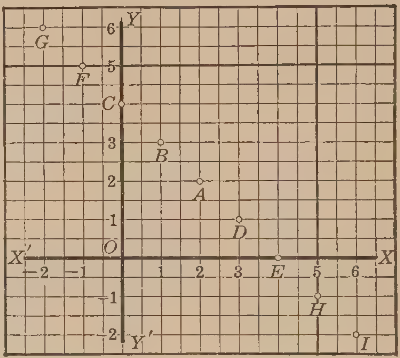
The point of intersection of the axes is called the **origin**.

The values of the x-distance and the y-distance of a point are often called the **coördinates** of the point.

Though not an absolute necessity, cross-ruled paper is a great convenience in all graphical work. Excellent results, however, can be obtained with ordinary paper and a rule marked in inches and fractions of an inch for measuring distances. Hence the graphical work which follows should not be omitted even though it is found inconvenient to obtain cross-ruled paper for class use.

EXAMPLE

Using $\frac{1}{5}$ inch for the unit of measure, locate points corresponding to the pairs of related numbers which follow: $A, (2, 2)$; $B, (1, 3)$; $C, (0, 4)$; $D, (3, 1)$; $E, (4, 0)$; $F, (-1, 5)$; $G, (-2, 6)$; $H, (5, -1)$; and $I, (6, -2)$.



Solution. According to VI, page 203, the x-distance of the point (2, 2) is 2 and its y-distance is 2. Hence to locate point A at (2, 2) we measure, according to IV, two units to the right of the origin on the x-axis and from that point two units upward.

To locate point F at $(-1, 5)$ we measure on the x-axis one unit to the left of the origin and from that point upward five units. Point F thus located corresponds to $(-1, 5)$.

Points for the other pairs of numbers given in the example should be located by the pupil. The correct positions for these points can be seen in the figure.

Locating points as in the example above is called **plotting points.**

EXERCISES

Draw two axes and plot the following points, using $\frac{1}{2}$ inch as the unit of measure.

1. (2, 3); (3, 1); (4, 2.5); and (1.5, 4).

2. $(4, -2)$; $(5.5, -3)$; $(6, -1)$; $(-1.5, 4)$; $(-1.8, 2)$; and $(-3, -3)$.

3. (3, 0); (5, 0); (1, 0); (0, 0); (0, 3); and (0, 4.5).

4. $(-2, 0)$; $(-5, 0)$; $(-1, 0)$; $(0, -3.5)$; and $(0, -4)$.

5. If the x-distance of a point is zero, where is the point located? Where is it located if both of its coördinates are zero?

92. The graph of an equation. On page 202 we computed several sets of values of x and y for the equation $x+y=4$. Later these points were plotted in locating A, B, C, D, E, F, G, H, and I of the example on the preceding page. It is evident from an inspection of their position that a straight line can be made to pass through all of the points there located. The line drawn through these points is said to be the **graph of the equation** $x+y=4$.

EXERCISES

1. Find and tabulate six pairs of values of x and y which satisfy the equation $2x+y=8$. Draw two axes and, using $\frac{1}{2}$ inch as the unit distance, plot each of the points. Are the six points in a straight line? Where do all the points lie whose x- and y-distances satisfy the equation $2x+y=8$? What, then, is the graph of the equation $2x+y=8$? Does $x=3$, $y=3$ satisfy this equation? Plot the point (3, 3). Is it on the graph of the equation? If the x- and y-distances of a point satisfy the equation $2x+y=8$, where is the point located? If the x- and y-distances of a point do not satisfy the equation $2x+y=8$, where is the point located?

Find and tabulate six pairs of values for x and y which satisfy each of the following equations. Use numbers not greater than 10. Use at least one negative value for x and one negative value for y. Then plot the six corresponding points. Can a straight line be drawn through the six points located for each exercise?

2. $2x+3y=6$.	**4.** $x+y=0$.	**6.** $2x=y$.
3. $4x-3y=12$.	**5.** $x-y=0$.	**7.** $y=3x$.

The preceding work should *convince* the student that the graph of an equation of the first degree in x and y is a *straight line.* This fact can be proved, but the student would not understand the proof were it given at present. Therefore it will be *assumed* that the graph of every linear equation in two variables is a straight line. And as a straight line is determined by *any two* of its points, in graphing a linear equation in two variables it will be sufficient to plot *any two points* whose x- and y-distances satisfy the equation, and then to draw through these two points a straight line. The two points most convenient to plot are usually the two in which the line cuts the axes. Occasionally these points come very close together, and consequently they will not determine accurately the position of the line. In such cases one should decide on two values of x rather far apart (such as 0 and 5, or 0 and -5) and compute the corresponding values of y. Two such points will fix the position of the line with sufficient accuracy.

If a line goes through the origin (as in Exercise 6 preceding), $x=0$, $y=0$ will do for one point, but a point not on the axes must be taken for the second one.

The essentials of the method of graphing a given linear equation in x and y are illustrated in the following example.

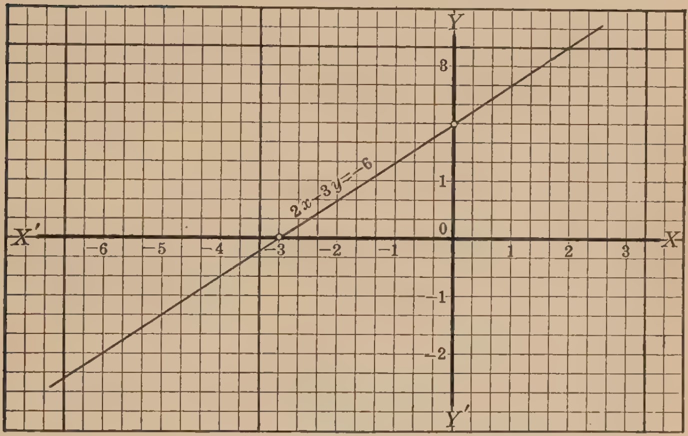
EXAMPLE

Graph the equation $2x-3y=-6$. In this equation, if $x=0$, $y=2$; and if $y=0$, $x=-3$. Here the point (0, 2) is on the y-axis in the adjacent figure, 2 units above the origin; and the point $(-3, 0)$ is on the x-axis, 3 units to the left of the origin. The straight line through these two points is the graph of $2x-3y=-6$.

The necessary work may be tabulated as follows:

$$2x - 3y = -6.$$

If $x =$	0	-3	2
then $y =$	2	0	$3\frac{1}{3}$



Check. If an error has been made in obtaining the value of x or y from the equation, or in plotting the values found, it can be quickly detected by plotting a third point, as $(2, 3\frac{1}{3})$, the values of whose x- and y-distances satisfy the equation. If this third point lies on the line determined by the first two points, the line has been correctly located; if it does not, an error has been made.

EXERCISES

Graph the following linear equations:

1. $x + y = 5.$
2. $x - y = 4.$
3. $x + 2y = 6.$
4. $4x + 3y = 12.$
5. $5x - 3y = 15.$
6. $2x + 3y = 10.$
7. $2x - y = 0.$
8. $x - 3y = 0.$
9. $x = 4.$

HINT. The equation $x = 4$ is equivalent to the equation $x + 0y = 4$. This last is satisfied by $x = 4$ and *any* value of y. Thus the pairs of values

(4, 3); (4, 6); (4, 0); (4, -2), etc., satisfy the equation $x + 0y = 4$. Plotting these points, it is evident that the required graph is a line parallel to the y-axis and 4 units to the right of it.

10. $x = -4$. **12.** $y = -3$. **14.** $x = 0$.

11. $y = 6$. **13.** $y = 0$. **15.** $x = 5$.

16. If a point is on a line, do the values of its x-distance and its y-distance satisfy the equation of the line?

17. If the values of the x-distance and the y-distance of a point satisfy the equation of a line, is the point located on the graph of the equation?

18. Is the point (4, 3) on the line whose equation is $2x - 3y = 12$? is (0, 6)? is (6, 0)?

19. Can you determine without reference to the graph itself whether the point (6, 5) is on any of the graphs of the equations in Exercises 1–9 above? If so, on which does it lie?

It should now be clear that:

The equation of a line is satisfied by the values of the x-distance and the y-distance of any point on that line.

Any point the values of whose x-distance and whose y-distance satisfy the equation is on the graph of the equation.

93. Graphical solution of linear equations in two variables. If we construct the graphs of the two equations $5x + 4y = 12$ and $x - 2y = 8$ as indicated in the figure on page 209, it is seen that for the point of intersection of the graphs x is 4 and y is -2. Since the point (4, -2) is on both graphs, these values should satisfy both equations. Substituting 4 for x and -2 for y in each equation, we get the identities $20 + (-8) = 12$, and $4 - (-4) = 8$. Thus the **graphical solution** of two linear equations consists in plotting the two equations and finding from the graph the value of x and the value of y at the point of intersection.

Since two straight lines *can intersect in but one point*, there can be but *one pair of values of x and y* which satisfies a pair of linear equations in two variables.

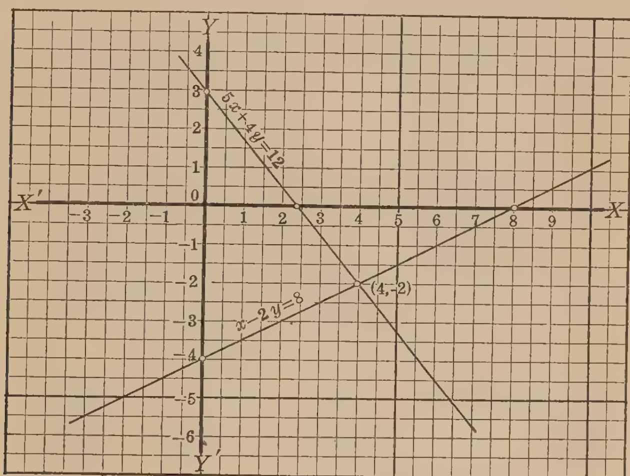
The necessary work is tabulated as follows:

$5x + 4y = 12$

If $x =$	0	$2\frac{2}{5}$	5
then $y =$	3	0	$-3\frac{1}{4}$

$x - 2y = 8$

If $x =$	0	8	2
then $y =$	-4	0	-3



EXERCISES

Solve graphically the following pairs of linear equations and verify by substituting in each pair of equations the x and y values of the point of intersection as obtained from their graphs:

1. $x + y = 5,$
   $2x + y = 8.$
2. $x + y = 7,$
   $x - y = 3.$
3. $3x + y = 6,$
   $x - 3y = -8.$
4. $x + y = 1,$
   $2x - y = 8.$
5. $3x - 2y = 10,$
   $x + 2y = 6.$
6. $x - 3y = 3,$
   $2x + y = -8.$

7. $4x - y = 6,$ $2x + y = 9.$

8. $2x + 5y = 10,$ $x + y = 5.$

9. $x - 2y = 8,$ $2x - y = 4.$

10. $x + 2y = 4,$ $x + 2y = 6.$

11. $x - 2y = 6,$ $2x - 4y = 9.$

12. $2x - y = 0,$ $x = 3y$

94. Graphical representation of statistics. Scientific data and numerical statistics from the business world are frequently exhibited with striking clearness and brevity by means of graphs. The form of the graph obtained in any case depends on the character of the relation between the plotted numbers. Sometimes the resulting graph is a smooth curve, and then again it may be an irregular continuous line made up of straight lines of various lengths.

The following graphs are types which occur with increasing frequency in the magazines and in the daily papers. Such graphs display data effectively, and inferences not otherwise apparent can often be drawn from them.

Biographical Note. *René Descartes.* One of the two or three most important advances ever made in mathematics was the discovery that algebraic equations could be represented geometrically. This great discovery was made by René Descartes (1596–1650), the French philosopher. Though never rugged in health, he took part in several campaigns when a young man, and it is said that during a weary winter spent in camp in Austria he first conceived the ideas that resulted in this important work. Though his writings read very differently from a modern book on the same subject, yet he developed all of the essentials of graphical representation. He saw that a letter, that is, a coördinate, might represent either a positive or a negative number, and so enforced upon mathematicians the conviction that negative integers are indeed numbers and that they are useful in algebraic operations. After his time they were not usually ruled out as absurd or impossible, as was commonly the case before. He also introduced the modern exponential notation, though he did not use negative or fractional exponents. To Descartes is due the use of the last letters of the alphabet for the unknown and the first letters for the known numbers. Thus he would have written the equation $x^3 - 8x + 16 = 40$ in the form $x^{3*} - 8x + 16 \infty 40$. Though the sign $=$ was used long before his time, he did not accept it. The asterisk he used to indicate that a certain power of the variable was lacking.

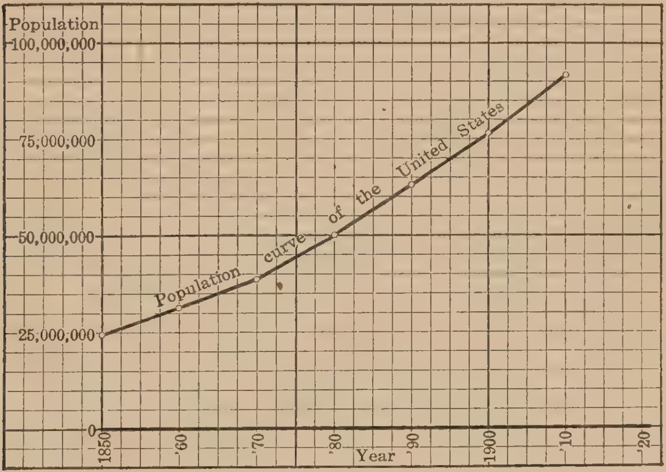
RENÉ DESCARTES

EXAMPLE 1

The census reports of the United States show that the population in millions for ten-year intervals was as follows:

Year	1850	1860	1870	1880	1890	1900	1910
Population in millions	23.2	31.4	38.6	50.2	62.6	76.0	92.0

The graphical representation of these statistics is given below. On the graph the population is measured parallel to the vertical axis, $\frac{1}{10}$ of an inch representing 5,000,000 people. The ten-year intervals are measured along the horizontal axis, with $\frac{1}{10}$ of an inch representing $2\frac{1}{2}$ years.



QUESTIONS ON EXAMPLE 1

1. What population does the graph indicate for 1885? for 1905?

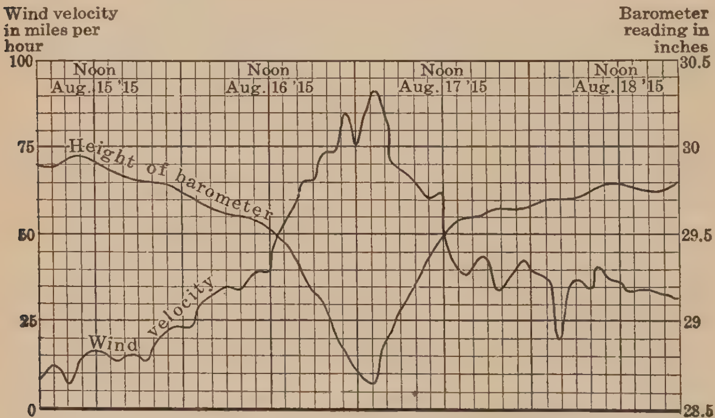
2. What cause may be assigned for the downward bend of the curve at 1870?

3. What population does the graph indicate for the United States in 1915? in 1920?

4. What causes might make the population of the United States in 1920 differ from the value indicated by the curve?

EXAMPLE 2

Graphical record of atmospheric pressure and wind velocity for the Galveston, Texas, storm period of August 15–18, 1915. The great storm occurred August 17.



QUESTIONS ON EXAMPLE 2

1. What was the velocity of the wind at midnight August 15? at midnight August 16?

2. At what times was the wind velocity 40 miles per hour? 80 miles per hour?

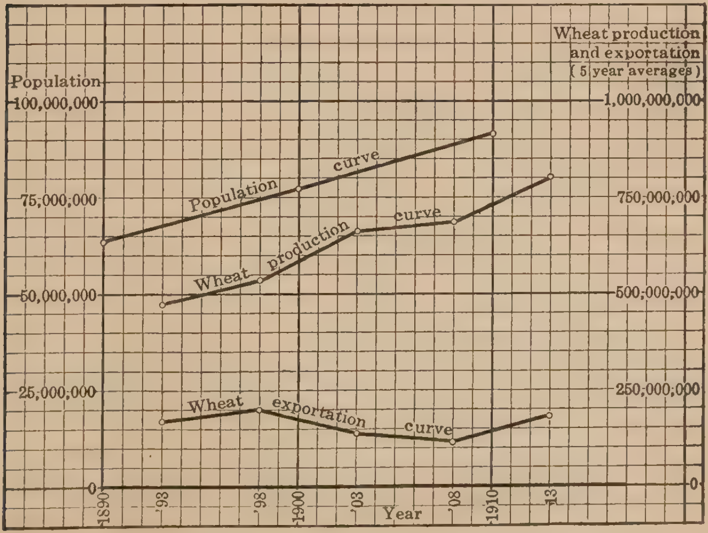
3. What was the reading for barometric pressure (height of the barometer) at midnight August 15? at midnight August 16? at 2 A.M. August 17?

4. What was the lowest reading of the barometer and the highest velocity of the wind? When did this occur?

EXAMPLE 3

In the following graph the upper curve shows in millions the population of the United States for 1890, 1900, and 1910. The middle and lower curves show the average yearly production and exportation respectively of wheat in millions of bushels (using the average for 5-yeaı intervals) for the years 1891–1915 inclusive. The general tendency of production and exportation is made more evident by plotting not yearly values but averages for five-year intervals, since by this method the high or low record for any one year is not emphasized unduly.

Intervals	1891–1895	1896–1900	1901–1905	1906–1910	1911–1915
Average yearly production	477	540	660	681	803
Average yearly exportation	167	197	140	116	188



QUESTIONS ON EXAMPLE 3

1. Estimate the population of the United States for 1892; for 1907; for 1912.

2. Compare the rate of increase in the production of wheat since 1901 with the rate of increase of the population.

3. Does the graph reveal any general tendency in the number of bushels of wheat which is exported? What tendency do you observe?

4. Does the graph indicate that the number of bushels exported will increase to any considerable extent? Explain.

EXERCISES ON THE GRAPHING OF STATISTICS

1. The records of the United States Weather Bureau of a certain city show hourly temperature records for a certain day in July. These records reported for the hours ending 1 A.M. to 12 P.M. inclusive, expressed in degrees, are as follows:

A.M., hour ending at	1	2	3	4	5	6	7	8	9	10	11	12
Temperature record .	84	83	80	80	79	77	77	79	84	88	89	93

P.M., hour ending at	1	2	3	4	5	6	7	8	9	10	11	12
Temperature record .	98	100	88	72	81	86	84	82	80	75	74	74

Graph the foregoing data.

A thunderstorm occurred on the day in which the above record was made. From the graph determine the time of the storm. How do you account for the sudden fall in temperature after 2 P.M.? for the sudden rise in the graph after 4 P.M.?

2. From reports gathered from several cities it has been estimated that for 100 children in school at the age of 8 the numbers

of pupils remaining in school (not counting those eliminated by death) for each year up to and including age 18 are as follows:

Age of pupils . .	8	9	10	11	12	13	14	15	16	17	18
Number in school .	100	100	100	98	97	88	70	47	30	16.5	8.6

Graph the foregoing data. Let $\frac{1}{2}$ inch represent one year on the horizontal axis and ten pupils on the vertical axis.

From the graph determine the years for which the tendency to leave school first becomes pronounced. Does the graph show a greater tendency on the part of 14-year-old and 15-year-old pupils to leave school than on the part of those of 12 and 13 years? How does the graph show this fact? Explain.

3. Estimates worthy of consideration are here given which show a contrast in the average weekly earnings for boys who leave school at the age of 14 (the usual end of the grammar-school period) with weekly earnings of boys who leave school at the age of 18 (the usual end of the high-school period).

Age	Boys who leave school at the age of 14 earn for the years indicated at the left the weekly wages given below:	Boys who leave school at the age of 18 earn for the years indicated at the left the weekly wages given below:
14	$4.00	0
16	$5.00	0
18	$7.00	$10.00
20	$9.50	$15.00
22	$11.00	$20.00
24	$12.00	$24.00
25	$13.00	$30.00

Graph the two sets of data on the same axes.

Determine from the graph which group of boys receives the more rapid increase in weekly wage.

4. The value in millions of dollars of the merchandise imported into and exported from the United States for the years 1914 to 1921 inclusive was as follows:

Year	1914	1915	1916	1917	1918	1919	1920	1921
Imports	1893	1674	2197	2659	2945	3095	5238	3654
Exports	2364	2768	4333	6290	5919	7232	8108	6516

Plot the foregoing data, using $\frac{1}{2}$ inch for one year on the horizontal axis and for 250 million dollars on the vertical axis.

How do you account for the decreased imports of 1915? the decreased exports of 1914? the increased exports of 1915?

5. The purchasing power of a dollar based on the wholesale price of a large number of commodities for the interval 1903 to 1921 is given in the following table:

Year	1903	1904	1905	1906	1907	1908	1909
Cents	100	100	94.3	90.3	90.3	91.8	82.6
Year	1910	1911	1912	1913	1914	1915	1916
Cents	79.5	80.9	78.2	78.9	79.4	77.8	64.0
Year	1917	1918	1919	1920	1921		
Cents	45.2	40.4	37.4	32.6	51.6		

Graph the foregoing data, showing the change in the purchasing power of a dollar through the given interval.

What tendency in the change in the purchasing power of a dollar does the graph indicate? Could more or less merchandise be bought for a given sum in 1910 or in 1920? Approximately how much more? A salary of $1500 in 1910 was equivalent to what salary in 1920?

CHAPTER XVIII

LINEAR SYSTEMS

95. Definitions. A simple or **linear** equation in several unknowns is one which may be put in such a form that

(*a*) no unknown appears in any denominator;
(*b*) only one unknown appears in any term;
(*c*) only the first power of any unknown is involved.

The following equations are linear: $2x+3y=1$; $4u+6v+2w=4$; the following equations are not linear: $2x+3xy-4y=1$; $4/u-6/v+2/w=2$; $x^2+3x-4y=(x+2)(2y-3)$.

In a previous chapter we have seen that a simple equation in one unknown has only one root; that is, the value of the unknown in such an equation is a constant.

Thus the value of the unknown in the equation $3x+4=7$ is the number 1, and this is the only root of the equation.

A linear equation in two unknowns is satisfied by an unlimited number of pairs of values for the two unknowns. A single equation in more than one unknown is called an **indeterminate** equation. The unknowns are sometimes called variables.

Thus the equation $x+y=10$ is satisfied by any pair of numbers whose sum is 10. Evidently, if one includes positive, negative, and fractional numbers, there is no limit to the number of pairs whose sum is 10. (See page 202.)

Two or more equations involving two or more unknowns are called a **system** of equations.

A system of equations all of which are satisfied by the same values of the unknowns is called a **simultaneous system** of equations.

The two equations $x + y = 9$ and $x - y = 1$ form a simultaneous system. The two numbers that satisfy both equations must be such that their sum is 9 while their difference is 1. These conditions are satisfied by $x = 5$, $y = 4$.

A set of values (one for each unknown) which satisfies an equation in two or more unknowns is sometimes called a solution of the equation; and a set which satisfies a system is often called a solution of the system. In this book, however, the word *solution* will be used to denote the *process of solving* either a single equation or a system. The values of the unknown which satisfy an equation in one unknown will be called *roots*, and a set of values for the unknowns satisfying an equation in two or more unknowns, or a system of such equations, will be called a **set of roots**.

ORAL EXERCISES

In Exercises 1–4 find the value of x corresponding to each of the values for y indicated at the right:

1. $x + y = 10$. $\qquad y = 1, 5, 6$.
2. $x - y = 1$. $\qquad y = 4, 10, 0$.
3. $x + 2y = 16$. $\qquad y = 0, 4, 8$.
4. $x - 2y = 0$. $\qquad y = 1, 4, -1$.

In Exercises 5–8 determine which of the pairs of numbers written at the right of each equation satisfies that equation. (The first number of a pair always denotes the value of x.)

5. $2x + 3y = 10$ $\qquad (2, 2)$; $(1, 3)$; $(8, -2)$.
6. $3x - y = 0$ $\qquad (2, 4)$; $(4, 2)$; $(-1, 3)$.

7. $y - x = 0$ (3, 4); (6, 6); (−1, −1).

8. $3y - 6 = 2x$ (1, 4); (−3, 0); (−3, −1).

In Exercises 9–12 find two pairs of numbers which satisfy each equation, and two other pairs which do not.

9. $x + 2y = 7$.

10. $x - 3y = 5$.

11. $2x + 3y = 16$.

12. $3y + 4x = 2$.

96. Solution by addition and subtraction. It was shown in the previous section that there is an unlimited number of sets of roots of a given linear equation in x and y, and that there is also an unlimited number of pairs of values of x and y which do not satisfy the equation. We now proceed to give an algebraic method of determining whether or not there is any common set of roots for two given linear equations. It turns out that there is usually one and only one such set of roots. This set may always be found by the method illustrated in the following examples.

EXAMPLES

1. Solve the system $\begin{cases} x + y = 4, & (1) \\ x - 2y = 1. & (2) \end{cases}$

Solution. Eliminate x first, thus:

(1) − (2), $\quad 3y = 3. \quad (3)$

(3) ÷ 3, $\quad y = 1. \quad (4)$

Substituting 1 for y in either (1) or (2), say in (1),

$$x + 1 = 4. \quad (5)$$

Solving (5), $\quad x = 3. \quad (6)$

Check. Substituting 3 for x and 1 for y in (1) and (2) gives the identities

$$3 + 1 = 4,$$

and

$$3 - 2 = 1.$$

2. Solve the system $\begin{cases} 5x + 3y = 1, & (1) \\ 3x - 2y = 12. & (2) \end{cases}$

Solution. Eliminate y first, thus:

$(1) \cdot 2,$ $\quad 10x + 6y = 2.$ $\quad (3)$

$(2) \cdot 3,$ $\quad 9x - 6y = 36.$ $\quad (4)$

$(3) + (4),$ $\quad 19x = 38.$ $\quad (5)$

$(5) \div 19,$ $\quad x = 2.$ $\quad (6)$

Substituting 2 for x in (2), $6 - 2y = 12.$ $\quad (7)$

Solving (7), $\quad y = -3.$ $\quad (8)$

Check. Substituting 2 for x and -3 for y in (1) and (2) gives

$$10 - 9 = 1,$$

and

$$6 + 6 = 12.$$

Either x or y could have been eliminated first. The multipliers necessary to eliminate x are 3 and 5, while the multipliers necessary to eliminate y are the *more convenient numbers* 2 and 3.

When the notation $(3) - (4)$ is used in a solution, it indicates the subtraction of the first member of equation (4) from the first member of equation (3), the subtraction of the second member of equation (4) from the second member of equation (3), and the writing of the two results as an equation. The process of adding the corresponding members of the two equations is indicated by writing $(3) + (4)$.

The notation $(3) \cdot 6$ indicates that both members of equation (3) are multiplied by 6, and $(3) \div 6$ indicates that both members of equation (3) are divided by 6.

With the meanings just explained it is customary to speak of the addition or the subtraction of two equations and of the multiplication or division of an equation by a number.

The method of the preceding solutions is stated in the

Rule. *If necessary, multiply the first equation by a number and the second equation by another number, such that the coefficients of the same unknown in each of the resulting equations will be numerically equal.*

If these coefficients have like signs, subtract one equation from the other; if they have unlike signs, add. Then solve the equation thus obtained.

Substitute the value just found, in the simplest of the preceding equations which contains both unknowns, and solve for the other unknown.

Check. *Substitute for each variable in the original equations its value as found by the rule. If the resulting equations are not obvious identities, simplify them until they become such.*

An attempt to solve by the rule the pair

$$\begin{cases} 3x - 6y = 40, & (1) \\ x - 2y = 8, & (2) \end{cases}$$

gives
$$3x - 6y = 40, \qquad (3)$$
$$3x - 6y = 24. \qquad (4)$$

(3) − (4), $0 = 16$, which is false.

This result indicates that (1) and (2) do not form a simultaneous system, but are **incompatible** equations.

The graphs of a pair of incompatible linear equations are parallel lines (see Exercise 10, p. 210).

An attempt to solve by the rule the system $\begin{cases} x + 2y = 8, \\ 3x + 6y = 24, \end{cases}$ gives $0 = 0$. Here the second equation divided by 3 gives the first. Therefore any set of roots of the first is a set of the second. If we choose to regard the two equations as really different, which is not at all necessary, we say that they have an infinite (unlimited) number of sets of roots. Two or more equations having this property constitute an **indeterminate system**.

EXERCISES

Solve the following systems and check results:

1. $\begin{matrix} x + y = 3, \\ x - y = 1. \end{matrix}$

2. $\begin{matrix} x + y = 7, \\ x - y = 1. \end{matrix}$

3. $\begin{matrix} 2x + y = 9, \\ x - y = 6. \end{matrix}$

4. $\begin{matrix} x + y = 5, \\ x - 3y = 1. \end{matrix}$

5. $\begin{matrix} x + y = -3, \\ x - y = 3. \end{matrix}$

6. $\begin{matrix} x + 2y = 7, \\ 5x - 2y = 11. \end{matrix}$

7. $2y + x = 4,$ $3y - x = 21.$

8. $7r - s = 2,$ $6r - s = 3.$

9. $3x - y = 3,$ $5x + 2y = 16.$

10. $4x + y = 2,$ $x - 2y = 5.$

11. $x + 2y = 1,$ $3x + 10y = 4.$

12. $5m - 3n = 0,$ $15m + 12n = 75.$

13. $x - 6y = 7,$ $12y - x = -1.$

14. $2x + 25y = 70,$ $3x = 10y + 10.$

15. $4p + q = 5,$ $p - 4q = 14.$

16. $12x + 5y = 6,$ $3x - 3y = 10.$

17. $3x - 4y = 14,$ $3y - 4x = -14.$

18. $4x + 3y = 5,$ $9y - 8x = 0.$

19. $6u + 8v = 26,$ $5u - 3v = 70.$

20. $11x + 7y = 111,$ $3y - 4x = 4.$

21. $5y + 3z = 37,$ $9z + 15y = 111.$

97. Solution by substitution. The method of solving a system of two linear equations by substitution is illustrated in the following example.

EXAMPLE

Solve the system $\begin{cases} 4x - 12y = 44, & (1) \\ 8x + 11y = 18. & (2) \end{cases}$

Solution. From (1), $$4x = 12y + 44, \quad (3)$$

or $$x = 3y + 11. \quad (4)$$

Substituting $3y + 11$ for x in (2),

$$8(3y + 11) + 11y = 18. \quad (5)$$

Simplifying, $$24y + 88 + 11y = 18. \quad (6)$$

Collecting, $$35y = -70, \quad (7)$$

or $$y = -2. \quad (8)$$

Substituting -2 for y in (4),

$$x = -6 + 11 = 5. \quad (9)$$

Check. Substituting 5 for x and -2 for y in (1) and (2) gives

$$20 + 24 \doteq 44,$$

and $$40 - 22 = 18.$$

The method of the preceding solution is stated in the

Rule. *Solve either equation for one unknown in terms of the other.*

Substitute this value in place of the unknown in the equation from which it was not obtained and solve the resulting equation.

Substitute the definite value just found, in the simplest of the preceding equations which contains both unknowns, and solve, thus obtaining a definite value for the other unknown.

Check. See page 221.

The method of substitution emphasizes the fact that the values of x and y which are sought are the same in both equations. Hence an expression for an unknown obtained from one equation is substituted for that unknown in the other equation. This method is useful when one of the unknowns can be expressed in terms of the other without using fractions or when simple fractions only are involved.

EXERCISES

Solve by the method of substitution and check results:

1. $x - 2y = 8,$ $3x + 2y = 8.$

2. $x - 2y = -1,$ $4x - y = 10.$

3. $14m - 2n = 1,$ $n - 6m = 0.$

4. $6x + 10y = 42,$ $2y = 3x.$

5. $5x + 3y = -1,$ $2x - 6y = 8.$

6. $x + y = 7,$ $2x + 3y = 17.$

7. $3x = 6y - 3,$ $y + 4x = 5.$

8. $5x + 10y = 25,$ $5x - 3y = 9.$

9. $20y - 3z = 1,$ $z - 6y = 0.$

10. $3x + 12 = 3 + y,$ $y = x + 1.$

11. $18 + 2k = h,$ $h + k = -9.$

12. $2x = 4y + 6,$ $7x + 3y = 4.$

98. Simultaneous equations containing fractions. The method of solving a system of two linear equations containing fractions is illustrated by the following example.

EXAMPLE

Solve the system
$$\begin{cases} \dfrac{8x}{3} - \dfrac{5}{6} = \dfrac{3y}{2}, & (1) \\[2ex] \dfrac{2x}{3} = \dfrac{y}{4} + \dfrac{7}{12}. & (2) \end{cases}$$

Solution. $(1) \cdot 6,$ $\quad 16x - 5 = 9y.$ $\quad$ (3)

Transposing in (3), $\quad 16x - 9y = 5,$ $\quad$ (4)

$(2) \cdot 12,$ $\quad 8x = 3y + 7.$ $\quad$ (5)

Transposing in (5), $\quad 8x - 3y = 7.$ $\quad$ (6)

The system (4) and (6) can now be solved by addition and subtraction.

As in the foregoing solution, it is usually best to clear the equations of fractions and write them in the form of (4) and (6) before attempting to eliminate one of the unknowns. Equations (4) and (6) are each in what is called the **general form** of a linear equation in two unknowns. This form is represented for all such equations by $ax + by = c$, where a, b, and c denote numbers, or known literal expressions.

EXERCISES

Solve the following systems and check results:

1. $\begin{cases} 4x + \dfrac{2y}{3} = \dfrac{26}{3}, \\[2ex] 3y - \dfrac{7x}{2} = -4. \end{cases}$

2. $\begin{cases} \dfrac{5x}{6} + \dfrac{y}{4} = 7, \\[2ex] \dfrac{2x}{3} - \dfrac{y}{8} = 3. \end{cases}$

3. $\begin{cases} x - \dfrac{3y}{5} = \dfrac{18}{5}, \\[2ex] 7x + \dfrac{8y}{3} = -16. \end{cases}$

4. $.4x + .9y = 5.7,$
$x - \frac{y}{2} = \frac{1}{2}.$

5. $\frac{x}{3} + \frac{y}{2} = \frac{13}{6},$
$2y - 3x = 0.$

6. $\frac{x}{3} + \frac{y}{4} = -\frac{7}{8},$
$\frac{y}{3} - \frac{x}{4} = \frac{11}{12}.$

7. $.04x + .3y = 1,$
$.5x - .25y = 4.5.$

8. $7x - 4y = 14,$
$\frac{2x+7}{2} - y = 4.$

9. $\frac{x}{3} + \frac{y}{4} = \frac{1}{6},$
$\frac{x-2}{5} - \frac{y+1}{2} = \frac{1}{2}.$

10. $\frac{2x+6}{5} + \frac{y+4}{5} = 3,$
$\frac{7x+1}{3} - \frac{11y-4}{7} = 4.$

11. $\frac{x+y}{2} - 8 = \frac{x-y}{3},$
$\frac{x-y}{4} + \frac{x+y}{3} = 11.$

12. $\frac{x}{6} - \frac{y}{4} = 1,$
$\frac{4+5x}{11} - \frac{3-2y}{5} + 4 = 0.$

13. $\frac{1}{x} + \frac{1}{y} = \frac{5}{6},$
$\frac{1}{x} - \frac{1}{y} = \frac{1}{6}.$

HINT. Solve without clearing of fractions, using addition and subtraction.

14. $\frac{1}{x} + \frac{1}{y} = \frac{4}{15},$
$\frac{3}{x} + \frac{2}{y} = \frac{19}{15}.$

15. $\frac{6}{x} + \frac{7}{y} + \frac{3}{2} = 0,$
$\frac{7}{x} - \frac{6}{y} = \frac{16}{3}.$

16. $\frac{5}{x} + 12 = 17,$
$\frac{2}{x} - 3y = 0.$

17. $\frac{x-y}{2} = \frac{25}{6} - \frac{x+y}{3},$
$\frac{x+y-9}{2} + \frac{x-y+6}{3} = 0.$

18. $\frac{x-2}{5} - \frac{y-10}{4} = \frac{10-x}{3},$
$\frac{y+2}{3} - \frac{2x+y}{16} = \frac{x+13}{8}.$

19. $\frac{x+2}{7} + 8 - 2x = \frac{x-y}{4},$
$2y - \frac{3x-2y}{3} = 3x + 4.$

20. $$\frac{2}{x-y}-\frac{3}{2x+3y-14}=0, \qquad (1)$$
$$2x-\frac{y}{4}=\frac{5}{4}. \qquad (2)$$

HINTS. Multiplying (1) by $(x-y)(2x+3y-14)$,

$$2(2x+3y-14)-3(x-y)=0, \qquad (3)$$

or $$x+9y=28. \qquad (4)$$

(2) · 4, $$8x-y=5. \qquad (5)$$

Now solve equations (4) and (5) by substitution and check as usual.

21. $$\frac{4x+y}{6x+y}=\frac{2}{5},$$
$$x+\frac{10y}{3}=\frac{71}{3}.$$

22. $$\frac{4x}{3y+2}=4,$$
$$\frac{2x}{4x-3y+1}=\frac{1}{4}.$$

23. $$\frac{3x-4y+2}{x-y}=3,$$
$$\frac{x+y}{x}=\frac{5}{3}.$$

24. $$\frac{x-3}{y+1}=\frac{x-6}{y-2},$$
$$\frac{x+y-5}{x-y+1}=3.$$

25. $$\frac{5x+3y-5}{3}=y+\frac{5}{3},$$
$$\frac{25x}{10x+2y}=1.$$

26. $$\frac{r+5}{r+1}=\frac{s+2}{s-2},$$
$$\frac{11r+4}{r+s+1}=4.$$

27. $$\frac{2}{x+y+2}=\frac{1}{3x+5y-5},$$
$$\frac{x}{4}+\frac{y}{2}=1.$$

28. $$\frac{4}{x-1}=\frac{3}{1-y},$$
$$\frac{5}{2y-5}=\frac{7}{2x-39}.$$

In the following problems the student should state *two* equations in *two* unknowns. Instead of using x and y, the *first letter* of the word denoting an unknown should be used to represent that unknown. Thus in Problem 5, p. 227, n should represent the number of nickels and q the number of quarters. The plan here suggested is

desirable for many reasons, and should be followed in all problems containing two or more unknowns unless the words denoting two of the unknowns begin with the same letter.

In solving problems like 1–12 on pages 46–47, there are really two unknowns involved; but one of the equations to which each of those problems leads is so simple that what amounts to the method of substitution was employed by expressing one unknown in terms of the other.

PROBLEMS

1. The difference of two numbers is 20 and their sum is 36. Find the numbers.

2. The quotient of two numbers is 12 and their sum is 39. Find the numbers.

3. Find two numbers whose difference is 28 and whose quotient is 5.

4. The value of a certain fraction is $\frac{3}{4}$. If 6 be added to the numerator and 12 to the denominator, the value of the resulting fraction is $\frac{2}{3}$. Find the fraction.

5. A collection of nickels and quarters, containing 63 coins, amounted to \$8.35. How many coins of each kind were there?

6. If $\frac{2}{3}$ be subtracted from the numerator and $\frac{4}{3}$ added to the denominator of a certain fraction, the value of the resulting fraction is $\frac{1}{4}$. The sum of the numerator and the denominator of the original fraction is 11. Find the fraction.

7. The difference between the numerator and the denominator of a certain proper fraction is 11. If $\frac{1}{2}$ be added to the numerator and $\frac{3}{4}$ be taken from the denominator, the value of the resulting fraction is $\frac{6}{19}$. Find the fraction.

8. Two weights balance when one is 12 inches and the other 8 inches from the fulcrum. If the second weight increased by 8 pounds is placed 6 inches from the fulcrum, the balance is maintained. Find the two weights.

9. Two weights balance when one is 10 inches and the other 8 inches from the fulcrum. If the first weight is decreased by 4 pounds, the other weight must be moved 2 inches nearer the fulcrum to balance. Find the weights.

10. In 20 years A will be twice as old as B. Ten years ago A was 8 times as old as B. Find the age of each now.

11. The perimeter of a rectangle is 184 feet and the length is 8 feet more than twice the width. Find the dimensions of the rectangle.

12. A part of \$1500 is invested at 6% and the remainder at 5%. The combined yearly income is \$81. Find the number of dollars in each investment.

13. A part of \$3000 is invested at $4\frac{1}{2}\%$ and the remainder at $3\frac{1}{2}\%$. The annual income from the $4\frac{1}{2}\%$ investment is \$3 less than double the annual income from the $3\frac{1}{2}\%$ investment. Find the number of dollars in each investment.

14. A part of \$5000 is invested at 4% and the remainder at 6%. The 4% investment yields \$16 less in 4 years than the one at 6% does in 3 years. Find the number of dollars in each investment.

15. Five rubles are worth 5 cents less than 10 marks, and 12 marks are worth 4 rubles and a dollar. Find the value of a ruble and of a mark in cents.

16. During the war the value of marks in New York fell so much that on Jan. 1, 1916, 6 marks were worth exactly as much as 5 marks were worth on Aug. 1, 1914. Between the two dates mentioned the value of 10 marks decreased by 40 cents. Find the value of a mark, in cents, at each date.

17. The sum of the two digits of a two-digit number is 9. If 45 be subtracted from the number, the result will be expressed by the digits in reverse order. Find the number.

Solution. Let $t =$ the digit in tens' place,
and $u =$ the digit in units' place.
Then $t + u = 9.$ (1)

But t standing in tens' place has *its numerical value multiplied by* 10. Therefore the number is represented by the binomial $10\,t + u$, and the number formed by the digits in reverse order is represented by the binomial $10\,u + t$.

Hence $10\,t + u - 45 = 10\,u + t.$ (2)
Simplifying (2), $t - u = 5.$ (3)
Solving (1) and (3), $t = 7$, and $u = 2$.
Hence the number is 72.

Check. $7 + 2 = 9$, $72 - 45 = 27$.

18. The sum of the digits of a two-digit number is 10. If 54 be added to the number, the result is expressed by the digits in reverse order. Find the number.

19. The tens' digit of a two-digit number is half the units' digit. If 36 be added to the number, the result is expressed by the digits in reverse order. Find the number.

20. If a two-digit number be divided by the sum of its digits, the quotient is 4. Twice the given number is 9 greater than the number expressed by the same digits in reverse order. Find the number.

21. If a two-digit number be increased by 4 and then the result be divided by the sum of its digits, the quotient is 3. If twice the number be divided by the tens' digit, the quotient is 29. Find the number.

22. If a two-digit number be divided by the sum of its digits, the quotient is 7. If the number formed by the digits in reverse order be divided by 3 plus the sum of the digits, the quotient is 3. Find the number.

The **reciprocal** of a number is a fraction of which 1 is the numerator and the number itself is the denominator. Thus $\frac{1}{2}$ and $\frac{1}{a}$ are the reciprocals of 2 and a respectively.

23. What is the reciprocal of 6? $7a$? $\frac{2}{3}$? $\frac{3}{4x}$? $2\frac{1}{2}$? $3\frac{2}{5}$?

24. The sum of the reciprocals of two numbers is $\frac{11}{28}$, and the difference of their reciprocals is $\frac{3}{28}$. Find the numbers.

25. The difference of the reciprocals of two numbers is $\frac{1}{24}$. The quotient of the greater number divided by the less is $\frac{16}{15}$. Find the numbers.

26. If 5 grams be taken from one pan of a balance and placed in the other, the sum of the weights in the first will be $\frac{1}{2}$ the sum of those in the second. But if 15 grams be taken from the second and placed in the first, the weights in the two pans will then balance. Find the weight in each pan at first.

27. A gives B \$30; then B has twice as much money as A. B then gives A \$150 and has left $\frac{1}{5}$ as much as A. How many dollars had each at first?

28. The circumference of the fore wheel of a carriage is $1\frac{1}{2}$ feet less than that of the rear wheel. The fore wheel makes as many revolutions in going 286 feet as the rear wheel does in going 325 feet. Find the circumference of each wheel.

29. A and B together can do a piece of work in $7\frac{1}{5}$ days. They work together for 3 days, and A finishes the job by himself in 7 days. How many days would each require alone?

30. A man rows 10 miles downstream in 2 hours and returns in 4 hours. Find the rate of the river and his rate in still water.

HINT. Let $x =$ the man's rate in still water in miles an hour, and $y =$ the rate of the river in miles an hour. Then his rate downstream is $x + y$ miles an hour, and upstream $x - y$ miles an hour.

31. A boat goes downstream 45 miles in 3 hours and upstream 15 miles in 3 hours. Find its rate in still water and the rate of the current.

32. The rate of a boat in still water is $7\frac{1}{2}$ miles an hour. It goes down the river from A to B in 12 hours. It returns one half the distance from B to A in 9 hours. Find the rate of the river and the distance from B to A.

33. A boat which runs 12 miles an hour in still water goes downstream from A to C in 7 hours. It returns upstream to B, 36 miles below A, in 5 hours. Find the distance from A to C and the rate of the stream.

99. Literal equations in two unknowns. Linear systems in which the unknowns have literal coefficients are solved by the method of addition and subtraction.

EXERCISES

In the following exercises consider a, b, c, and d, as known numbers; solve for the other letters involved and check.

1. $x + y = 3a,$
$x - y = a.$

2. $5x + 4y = 17a,$
$8x + y = 11a.$

3. $3x - y = 10b,$
$4x + 9y = 3b.$

4. $5x - 4y = 10a - 4,$
$x - 2ay = 0.$

5. $3p + 4q - a = p - q + 22a,$
$p + a - q = 3p - 2q + 8a.$

6. $3x + 4y = 6c,$
$\frac{x}{c} - \frac{3y}{4c} = 2.$

7. $\frac{2x}{5} + \frac{2y}{3} = \frac{6a}{5},$
$\frac{y}{2} + \frac{x}{2} = a.$

8. $7.5m + 3n = 6b,$
$.25m + .5n = 0.$

9. $3x - 3y - 4a = x - b,$
$x + y + a + b = 2(a + b).$

10. $\frac{x}{2a} - \frac{2y}{a} = -10,$
$\frac{3x}{a} - \frac{7y}{4a} = \frac{3}{2}.$

11. $x + y = a,$
$x - y = b.$

12. $ax + 3y = 3 + 6a,$
$a^2x + y = 5a.$

13. $cx + y = 3,$
$c(y - 3) = x.$

14. $4x - 3y = 12(a - b),$
$3x - 2y = 9a - 8b.$

15. $4x + 2y = a,$
$2x + 4y = b.$

16. $2x - ay = b,$
$8x - by = 4a.$

17. $\frac{1}{x} - \frac{1}{y} = a,$
$\frac{3}{x} - \frac{2}{y} = b.$

18. $\frac{a}{x} + \frac{b}{y} = c,$
$\frac{a}{x} - \frac{b}{y} = d.$

19. $ax + by = a + b,$
$ax - by = a - b.$

20. $ax - by = c,$
$3ax - 2by = 4c.$

21. $ax + by = c,$
$hx + ky = m.$

GENERAL ORAL EXERCISES

1. If one book costs a dollars, what will $c + d$ books cost?

2. If a books cost b dollars, what will one book cost? c books?

3. What is the perimeter of a rectangle whose length is a and whose width is b? What is its area?

4. What is the perimeter of a rectangle whose length is $4a$ and whose width is b? What is its area?

5. The base of a triangle is $a + b$. The altitude is $a - b$. Find the area.

6. The base and the altitude of a triangle are each equal to $x - 2y$. Find the area.

7. The area of a triangle is k. The base is b. Find the altitude.

8. If x denotes A's age now, what does $x - 8$ denote? $x + 5$? What does the equation $x + 5 = 2(x - 8)$ signify?

9. Forty men pay d dollars each as dues to a society. The expenses of the society are n dollars. How many dollars are left in the treasury? How would you interpret the result if n is greater than $40\,d$?

10. If 10 apples can be bought for x cents, how many can be bought for y cents?

11. If a apples can be bought for 25 cents, how many can be bought for c cents?

12. A farmer has grain enough to last one horse d days. How long will this grain last k horses?

13. A farmer has grain enough to last h horses d days. How long will it last k horses?

GENERAL PROBLEMS

1. The altitude of a triangle is a inches and the base is 10 inches. If 2 inches be taken from the altitude, by how much must the base be increased so that the area will be the same as before?

HINT. Let $x =$ the increase of the base in inches.

Then
$$\frac{10\,a}{2} = \frac{(a-2)(10+x)}{2}.$$

2. The altitude of a triangle is a feet, the base is b feet. The altitude is increased h feet and the base decreased so that the area is the same as before. How many feet are taken from the base?

3. The sum of two numbers is s and their difference is d. Find the numbers.

4. The first of two numbers is a times the second, and the first minus the second equals b. Find the numbers.

5. The sum of two numbers is b, and the quotient of the first divided by the second equals a. Find the numbers.

6. If a be added to the numerator of a certain fraction, the value of the resulting fraction is 2. If b be added to the denominator, the value of the resulting fraction is 1. Find the fraction.

HINT. Let $\frac{n}{d}$ = the fraction. Then $\frac{n+a}{d}=2$, and $\frac{n}{d+b}=1$.

7. If the numerator of a certain fraction be increased by 1, the value of the resulting fraction is x. If the denominator of the fraction be decreased by 2, the value of the resulting fraction is y. Find the numerator and the denominator.

8. The value of a certain fraction is b. If 2 be added to the numerator, the value of the resulting fraction is c. Find the numerator and the denominator.

9. A boy who weighs a pounds and one who weighs b pounds balance at the opposite ends of a teeter board whose length is l feet. How far is the fulcrum from each boy?

10. A certain number of books at 80 cents each and another number at \$1.10 each cost together h dollars. If the prices of the books had been interchanged, the total cost would have been k dollars. Find the number of each kind.

11. Two books cost c dollars. The first cost d cents more than the second. Find the cost of each.

12. A and B have k dollars in all. If A gives h dollars to B they have equal sums. How many dollars had each at first?

13. If A gives h dollars to B, they will have equal sums. If B gives k dollars to A, A will have twice as much as B. How many dollars has each?

14. If A gives \$10 to B, B will have h dollars more than A. But if B gives k dollars to A, A will have three times as much as B. How many dollars has each?

15. A and B have together \$40. A gives h dollars to B, after which B gives k dollars to A. Then they have equal sums. How many dollars had each at first?

16. A gives r dollars to B and then has $\frac{1}{2}$ as much money as B. Then B gives \$8 to A and has left $\frac{3}{4}$ as much money as A. How many dollars had each at first?

17. A part of \$1000 is invested at $a\%$ and the remainder at $b\%$. The yearly income from both investments is c dollars. How many dollars are there in each investment?

18. A portion of x dollars is invested at 5% and the remainder at 4%. The yearly income is y dollars. How many dollars are there in each investment?

19. A works three times as fast as B. Together they can do a piece of work in c days. How many days would each require alone?

HINT. Let a and b denote the number of days required by A and B respectively to do the work alone.

Then $$3a = b, \text{ and } \frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

20. A works h times as fast as B. Together they can do a piece of work in 4 days. How many days would each require alone?

21. A and B together can do a piece of work in d days. A can do $\frac{2}{3}$ of the work in 6 days. How many days does each require alone?

22. A and B together can do a piece of work in 5 days. A can do $\frac{2}{5}$ of the work in k days. How many days does each require alone?

23. B requires twice as much time as A to do a piece of work which they can do together in n days. How many days does each require alone?

24. A and B together can do a piece of work in p days. A works q times as fast as B. How many days does each require alone?

100. Linear systems in three unknowns. The method of obtaining the set of roots of a system of linear equations in three unknowns is illustrated in the following:

EXAMPLE

$$\text{Solve the system}\begin{cases} 4x - 2y + z = 9, & (1)\\ 3x + y + 2z = 13, & (2)\\ 2x + 3y - 3z = -2. & (3)\end{cases}$$

Solution. Eliminate one unknown, say z, between (1) and (2), thus:

(1) · 2, $\quad 8x - 4y + 2z = 18. \quad$ (4)

(2), $\quad 3x + y + 2z = 13. \quad$ (5)

(4) − (5), $\quad 5x - 5y = 5, \quad$ (6)

or $\quad x - y = 1. \quad$ (7)

Now eliminate z between (1) and (3), as follows:

(1) · 3, $\quad 12x - 6y + 3z = 27. \quad$ (8)

(3), $\quad 2x + 3y - 3z = -2. \quad$ (9)

(8) + (9), $\quad 14x - 3y = 25. \quad$ (10)

Solving (7) and (10) we obtain $\quad x = 2,\ y = 1.$

Substitute 2 for x and 1 for y in (2),

$$6 + 1 + 2z = 13. \qquad (11)$$

Solving (11), $\quad z = 3. \quad$ (12)

Check. Substituting 2 for x, 1 for y, and 3 for z in (1), (2), (3),

$$8 - 2 + 3 = 9, \quad \text{or} \quad 9 = 9.$$
$$6 + 1 + 6 = 13, \quad \text{or} \quad 13 = 13.$$
$$4 + 3 - 9 = -2, \quad \text{or} \quad -2 = -2.$$

The foregoing example illustrates the

Rule. *Decide from an inspection of the coefficients which unknown is most easily eliminated.*

Using any two equations, eliminate that unknown.

With one of the equations just used, and the third equation, again eliminate the same unknown.

The last two operations give two equations in the same two unknowns. Solve these two equations by the rule (pp. 220–221).

Substitute the two values found in the simplest of the original equations and solve for the third unknown.

Check. *Substitute the values found in each of the original equations and simplify results.*

EXERCISES

Solve for the unknowns involved:

1. $x + y + z = 9,\quad x - y - z = -3,\quad x + y - z = 5.$

2. $x + y + z = 3,\quad 2x + y - z = 6,\quad x - y - 2z = 0.$

3. $x + y + z = 0,\quad 3x + 2y + 3z = -1,\quad x - y - 2z = -8.$

4. $3A + B - C = -8,\quad A - 4B + 2C = 9,\quad 2A + 3B + 3C = 13.$

5. $3x - 2y + 4z = 9,\quad 2x + 3y - 2z = -3,\quad 5x + 2y + 3z = 6.$

6. $6h + 5k + 4m = 9,\quad 4h + 6k + 5m = 5,\quad 5h + 4k + 6m = 16.$

7. $x + 3y + 2z = 17,\quad y - 4z = -5,\quad x + 2y = -8.$

8. $3x - y - 2z = -2,\quad 6x + z = 4,\quad 3y - 4z = -11.$

9. $x + y = 1,\quad y + z = 3,\quad z + x = 8.$

10. $2p_1 - 3p_2 = 4,\quad 3p_1 + p_3 = 5,\quad p_2 - 2p_3 = 2.$

11. $5x - 7y + 4z = 3,\quad 6x + 3y - 5z = -3,\quad 4x + 6y + 3z = 25.$

12. $3A_1 + 2A_2 + 4A_3 = 9,\quad A_1 - A_2 - 2A_3 = 3,\quad A_1 - 3A_2 + 2A_3 = 2.$

13. $\frac{1}{x} + \frac{1}{y} - \frac{1}{z} = 1,\quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{2}{3},\quad \frac{1}{x} - \frac{1}{y} + \frac{1}{z} = 0.$

HINT. Solve without clearing of fractions.

14. $\dfrac{2}{p}+\dfrac{10}{q}-\dfrac{3}{r}=-3,$
$\dfrac{4}{p}+\dfrac{5}{q}+\dfrac{6}{r}=15,$
$\dfrac{1}{p}+\dfrac{5}{q}-\dfrac{1}{r}=-\dfrac{1}{2}.$

15. $\dfrac{2}{x}+\dfrac{3}{z}=\dfrac{34}{3},$
$\dfrac{3}{x}+\dfrac{4}{y}=24,$
$\dfrac{4}{z}+\dfrac{5}{y}=28.$

PROBLEMS

1. Find three numbers of which the sum of the first and second is 54, the second and third 65, and the first and third 59.

2. The sum of three numbers is 70. The sum and the quotient of two of them are 45 and 5 respectively. Find the numbers.

3. The perimeter of a triangle is 60 feet. Two of its sides are equal, and the third side is 6 feet longer than either of the first two. Find the length of each side.

4. The sum of two sides of a triangle is 53 feet and their difference is 9 feet. The perimeter of the triangle is 72 feet. Find the length of each side.

5. The sum of the two sides of a triangle which meet at one vertex is 41 feet, at another vertex 46 feet, and at the third vertex 57 feet. Find the length of each side.

6. The sum of three numbers is 24. The quotient of two of them is 3, and the sum of these two divided by the third is 5. Find the numbers.

Fact from Geometry. The sum of the three angles of any triangle (each angle being measured in degrees) is 180 degrees.

7. Two of the angles of a triangle are equal, and their sum is equal to the third. Find the number of degrees in each angle.

8. Angle A of a triangle is 12 degrees greater than angle B, and angle B is 21 degrees greater than angle C. How many degrees are there in each?

9. The sum of two angles of a triangle is 30 degrees more than the third, and the third is 15 times the difference of the first two. How many degrees are there in each?

10. A and B together can do a piece of work in 6 days, A and C in 8 days, and B and C in 12 days. Find the time required by each alone and by all together.

11. Two pumps together can fill a tank in 6 hours. The first of these and a third together can fill the tank in 8 hours. All three together can fill the tank in 4 hours. Find the number of hours required by each alone.

12. The sum of the digits of a three-digit number is 19. The units' digit exceeds the tens' digit by 3. If 495 be added to the number, the result is expressed by the digits in reverse order. Find the number.

13. If the tens' and units' digits of a three-digit number be interchanged, the resulting number is 54 less than the original number. If the tens' and hundreds' digits be interchanged, the resulting number is 360 more than the original number. The sum of the digits is 11. Find the original number.

NOTE. Perhaps the student wonders whether a linear equation in three unknowns has a graphic representation. It may partially satisfy his curiosity to say that by means of three axes at right angles to each other such a representation, though beyond the scope of this book, is possible. Further, the points whose x, y, and z values satisfy the equation lie in a flat surface called a plane. Two such surfaces may intersect in a straight line, and the system of two equations which the surfaces represent is satisfied by the x, y, and z values of any point on this line. If three such surfaces intersect in a single point, the system which the surfaces represent is satisfied by the x, y, and z values of this point. In the systems of equations in three unknowns on page 237 the student is really finding the coördinates of the point of intersection of three planes.

Since space has but three dimensions, this method of representation of linear equations in two or three unknowns cannot be extended to equations containing four or more unknowns.

CHAPTER XIX

SQUARE ROOT

101. Square root of algebraic expressions. The square root of $t^2 + 2tu + u^2$ is $\pm(t + u)$.

A study of this form will enable us to extract the square root of any polynomial. Obviously the square root of t^2 (the first term of the trinomial) is t, the first term of the root. If t^2 is subtracted from the trinomial, the remainder is $2tu + u^2$. The next term of the root (u) can be found by dividing the first term of the remainder ($2tu$) by $2t$ (twice that term of the root already found).

The work may be arranged thus:

$$\begin{array}{lr|l}
 & t^2 + 2tu + u^2 & \underline{t + u} \\
 & t^2 & \\
\hline
\text{Trial divisor,} & 2t & 2tu + u^2 \\
\text{Complete divisor,} & 2t + u & 2tu + u^2 = (2t + u)u
\end{array}$$

Therefore the required roots are $\pm(t + u)$.

The foregoing process is easily extended to extracting the square root of the polynomial $4x^4 - 20x^3 + 37x^2 - 30x + 9$, whose square root contains *three* terms, as follows:

$$\begin{array}{lr|l}
 & 4x^4 - 20x^3 + 37x^2 - 30x + 9 & \underline{2x^2 - 5x + 3} \\
(2x^2)^2 = & 4x^4 & \\
\text{First trial divisor, } 2 \cdot 2x^2 = 4x^2 & & -20x^3 + 37x^2 \\
\text{First complete divisor, } 4x^2 - 5x & & -20x^3 + 25x^2 = (4x^2 - 5x)(-5x) \\
\text{Second trial divisor,} & & \\
\quad 2(2x^2 - 5x) = 4x^2 - 10x & & 12x^2 - 30x + 9 \\
\text{Second complete divisor, } 4x^2 - 10x + 3 & & 12x^2 - 30x + 9 = (4x^2 - 10x + 3)3
\end{array}$$

Therefore the required roots are $\pm(2x^2 - 5x + 3)$.

The term $2x^2$ was obtained by taking the square root of $4x^4$; the second term, $-5x$, by dividing $-20x^3$ by the first trial divisor, $4x^2$; and the third term, 3, by dividing $12x^2$ by $4x^2$, the first term of the second trial divisor.

The method just illustrated may be stated in the

Rule. *Arrange the terms of the polynomial according to descending powers of some letter in it.*

Extract the square root of the first term. Write the result (with plus sign only) as the first term of the root and subtract its square from the given polynomial.

Double the root already found for the first trial divisor, divide the first term of the remainder by it, and write the quotient as the second term of the root.

Annex the quotient just found to the trial divisor, making the complete divisor; multiply the complete divisor by the second term of the root and subtract the product from the last remainder.

If terms of the polynomial still remain, double the root already found for a trial divisor, divide the first term of the trial divisor into the first term of the remainder, write the quotient as the next term of the root, form the complete divisor, and proceed as before until the process ends, or until the required number of terms of the root have been found.

Inclose the root thus found in a parenthesis preceded by the sign $\pm$.

NOTE. The process of extracting the square root of numbers was familiar to mathematicians long before they knew how to find the square root of polynomials. This is consistent with the fact that the development of the methods of performing operations on literal number symbols generally followed and grew out of the similar operations on numerals. The application of the rules for extracting the square root of numbers to that of polynomials is generally

ascribed to Recorde (1510–1558), who was the author of the earliest English work on algebra that we know. This book, which bears the title "The Whetstone of Wit," gives an accurate idea of the algebraic knowledge of the time, and had a very wide influence.

EXERCISES

Obtain one square root of:

1. $a^2 + 6a + 9.$
2. $x^2 + 10ax + 25a^2.$
3. $x^4 + 16 - 8x^2.$
4. $4a^2x^2 + 4x^4 + a^4.$
5. $a^4 + 3a^2 + 2a^3 + 2a + 1.$
6. $24x^2 - 32x + 16 + x^4 - 8x^3.$
7. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$
8. $21c^2 + c^4 + 20c - 10c^3 + 4.$
9. $n^6 + 9n^2 + 10n^3 + 25 - 6n^4 - 30n.$
10. $c^4 - 12c + 9c^2 + 4 + 4c^2 - 6c^3.$
11. $5a^4 + 12a^5 + 16 - 23a^2 + 4a^6 + 8a - 22a^3.$
12. $c^4 - 4c^3a + 6c^2a^2 - 4ca^3 + a^4.$
13. $30xy^3 + 25y^4 - 11x^2y^2 - 12x^3y + 4x^4.$
14. $-36a^4x + 36a^2x^2 + 9a^6 - 24a^3x^2 + 16x^4 + 48ax^3.$
15. $9c^4 - 2a^2b^2c^2 + 4a^3b^3c + a^4b^4 - 12abc^3.$
16. $2a^2xc^3 - 4xc^3 - 4a^2x^2 + 4x^2 + c^6 + a^4x^2.$
17. $4 - \dfrac{4c^2}{5} + \dfrac{c^4}{25}.$
18. $\dfrac{9}{a^2} + \dfrac{a^2}{4} - 3.$
19. $x^4 - 4x^3 + 5x^2 - 2x + \frac{1}{4}.$
20. $a^4 + 4a^3 + 4a^2 - \dfrac{4a^2}{3} - \dfrac{8a}{3} + \dfrac{4}{9}.$
21. $x^4 - x^3 + \dfrac{x^2}{4} + \dfrac{2x^2}{3} - \dfrac{x}{3} + \dfrac{1}{9}.$
22. $\dfrac{25m^4}{4} - \dfrac{127m^2}{18} - 2m + \dfrac{10m^3}{3} + \dfrac{9}{4}.$

102. Square root of arithmetical numbers. Since $1 = 1^2$, and $81 = 9^2$, a one-digit or a two-digit square has only *one* digit in its square root.

And as $100 = 10^2$, and $9801 = (99)^2$, a three-digit or a four-digit square has *two* digits in its square root.

Also $10{,}000 = 100^2$, and $998{,}001 = (999)^2$; hence a five-digit or a six-digit square has *three* digits in its square root.

These examples illustrate the relation between the number of digits in a number and the number of digits in its square root. They also suggest a method of obtaining the first digit in the square root of any number.

For example, take the four numbers $78'43'56$, $7'84'35$, $.98'01$, and $.03'27'4$. Beginning at the decimal point in each, point off periods of two digits each, as indicated. Any period incomplete on the right, as in $.03'27'4$, should be completed by annexing one zero; thus, $.03'27'40$. Now the first digit in the square root is the greatest integer whose square is less than or equal to the left-hand period. This is true whether the latter contains *two* digits or *one*. Hence the first digit in the square root of $78'43'56$ is 8, in the square root of $7'84'35$ is 2, in the square root of $.98'01$ is 9, and in the square root of $.03'27'40$ is 1.

Moreover, the number of digits in the square root of a perfect square is equal to the number of periods, provided a *single digit* remaining on the left is called a period.

Just how t and u are involved in the square of $(t + u)$, or $t^2 + 2\,tu + u^2$, is obvious on inspection, because the parts t^2, $2\,tu$, and u^2 cannot be united into one term. In the square of an arithmetical number, however, the parts are united. Thus $(53)^2 = (50 + 3)^2 = 2500 + 300 + 9 = 2809$. Now it is clear how 50 and 3 are involved in $2500 + 300 + 9$, but it is not plain from 2809 alone. Pointing off, however,

enables us to discover at once the first digit, 5, which is equivalent to 5 tens, or 50. With the exception of pointing off, the method of extracting the square root of an arithmetical number does not differ greatly from the method of extracting the square root of an algebraic expression. In fact, the formula, the square root of $t^2+2\,tu+u^2=\pm(t+u)$, can be used to explain the two processes.

If t denotes the tens and u the units, $t^2+2\,tu+u^2$ is closely related to $2500+300+9$, t^2 being 2500, or $(50)^2$; u^2 being 9, or 3^2; and $2\,tu$ being $2\cdot 50\cdot 3$. Therefore the process of extracting the square root of 2809 may be based on these relations and the work arranged as follows:

$$\begin{array}{lll}
 & 2809 & \underline{|50+3} \\
t^2=(50)^2 & 2500 & \\
2t=2\cdot 50=100 & |\;309 & \\
2t+u=100+3 & |\;\underline{309=(100+3)\,3=(2t+u)u=2tu+u^2} &
\end{array}$$

Therefore $\pm$ 53 are the two square roots of 2809.

If the number has three digits in its square root, the work and explanations may be arranged thus:

$$\begin{array}{ll}
 & 1'74'24\;\underline{|100+30+2} \\
t^2=(100)^2 & \underline{1\,00\,00}=10 \text{ tens squared} \\
\text{First trial divisor,} & \\
\quad 2\,t=2\cdot 100=200 & |\;74\,24 \\
\text{First complete divisor,} & \\
2\,t+u=200+30=230 & |\;\underline{69\,00}=(2\cdot 10 \text{ tens}+30 \text{ units})\,30 \\
\text{Second trial divisor,} & \\
\quad 2\,t=2\cdot 130=260 & |\;5\,24 \\
\text{Second complete divisor,} & \\
2\,t+u=260+2=262 & |\;\underline{5\,24}=(2\cdot 13 \text{ tens}+2 \text{ units})\,2
\end{array}$$

Therefore $\pm\,132$ are the square roots of 17,424.

When the method and reasons for the process have become familiar, the work may be shortened by omitting the explanations and unnecessary zeros as follows:

```
     28'09|53                1'74'24|132
     25                      1
 103| 309                 23| 74
    | 309                   | 69
                          262| 524
                             | 524
```

The method of the two preceding solutions is the one commonly used for extracting the positive square root of a number. For it we have the

Rule. *Begin at the decimal point and point off as many periods of two digits each as possible: to the left if the number is an integer; to the right if it is a decimal; to both the left and the right if the number is part integral and part decimal.*

Find the greatest integer whose square is equal to or less than the left-hand period, and write this integer for the first digit of the root.

Square the first digit of the root, subtract its square from the first period, and annex the second period to the remainder.

Double the part of the root already found for a trial divisor, divide it into the remainder (omitting from the latter the right-hand digit), and write the integral part of the quotient as the next digit of the root.

Annex the root digit just found to the trial divisor to make the complete divisor, multiply the complete divisor by this root digit, subtract the result from the dividend, and annex to the remainder the next period for a new dividend.

Double the part of the root already found for a new trial divisor and proceed as before until the desired number of digits of the root have been found.

After extracting the square root of a number involving decimals, point off one decimal place in the root for every decimal period in the number.

Check. *If the root is exact, square it. The result should be the original number. If the root is inexact, square it and add to this result the remainder. The sum should be the original number.*

Sometimes in using a trial divisor we obtain too great a quotient for the next digit of the root. This happens in obtaining the second digit of the square root of 32,301, where 2 into 22 gives 11. Obviously 10 and 11 are both impossible. If 9 is tried we get $9 \cdot 29$, or 261, which is greater than 223. Similarly, 8 is too great. But $7 \cdot 27 = 189$, which is less than 223. Therefore 7 is the second digit of the root.

```
  3′23′01 | 1
  1
2 | 223
```

Occasionally the trial divisor gives a quotient less than 1. This indicates that the required root digit is 0, which should be written in the root and the work continued as usual. An instance of this kind occurs in finding the second digit in the square root of 9′42.49. The quotient of $4 \div 6$ is $\frac{2}{3}$, which is not an integer. Therefore the second digit of the root is 0. Then the next period, 49, should be brought down. The new trial divisor will be 60, which will give 7 as the third digit of the root. The work can easily be completed, giving 30.7 as the square root.

```
  9′42.49 | 30
  9
6 | 42
```

An attempt to extract the square root of 3 by annexing decimal periods of zeros and applying the rule becomes a never-ending process.

The number 3 has no exact square root, and no matter how far the work is carried, there is no final digit. As the work stands, we know that the square root of 3 lies between 1.732 and 1.733.

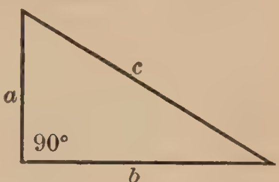
```
       3.00′00′00 | 1.732
       1
27   | 200
     | 189
343  | 1100
     | 1029
3462 | 7100
     | 6924
```

EXERCISES

Obtain the positive square root, to three decimal places, of:

1. 4489.	**4.** 6241.	**7.** 24649.	**10.** 165680.
2. 5184.	**5.** 9216.	**8.** 43436.	**11.** 223729.
3. 5329.	**6.** 16129.	**9.** 53361.	**12.** 328329.

Fact from Geometry. In the adjacent right triangle, $a^2 + b^2 = c^2$; the sides a and b, which form the right angle, are called the **legs**; and c, the side opposite the right angle, is called the **hypotenuse.**



If leg a is 8 and leg b is 15, then substituting in $a^2 + b^2 = c^2$ gives $64 + 225 = c^2$. Whence $289 = c^2$ and $c = \pm 17$.

Since -17 is not a practical answer, it is rejected.

In Exercises 13–16 find the hypotenuse and the area of a right triangle whose legs are:

13. 63 and 16.	**15.** 104 and 153.
14. 48 and 55.	**16.** 645 and 812.

In Exercises 17–19 find the other leg and the area of a right triangle in which the hypotenuse and one leg are respectively:

17. 109 and 91. **18.** 2.57 and .32. **19.** 2.05 and 1.87.

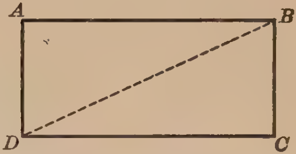
Extract the square root in Exercises 20–23 inclusive to four decimals, and in Exercises 24–28 inclusive to three decimals.

A common fraction or the fractional part of a mixed number should be reduced to a decimal before extracting the square root, unless the root is seen to be exact.

20. 6.4271.	**23.** .00321.	**26.** $2\frac{3}{5}$.
21. 884.3.	**24.** 3.	**27.** $2\frac{3}{7}$.
22. .0869.	**25.** 6.	**28.** $\frac{3}{17}$.

(In the following find all inexact answers to two decimals.)

In rectangle $ABCD$ line DB is called a **diagonal.**



29. Find the diagonal of a rectangle whose adjacent sides are 28 feet and 195 feet respectively.

30. One diagonal of a rectangle is 409 and one side is 391. Find the other side and the area.

31. One diagonal of a rectangle is 533 and one side is 92. Find the perimeter of the rectangle.

32. A rectangle is 7 yards longer than it is wide. Its perimeter is 138 feet. Find one diagonal.

33. One diagonal of a square is 74 meters. Find the side.

34. The side of a square is 52 inches. Find one diagonal.

35. One leg of a right triangle is 10. The hypotenuse is twice the other leg. Find the hypotenuse and the other leg.

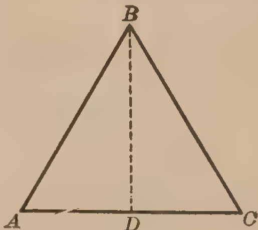
36. The hypotenuse of a right triangle is three times one leg and the other leg is 16. Find the sides.

37. A rectangle is 2.4 times as long as it is wide. One diagonal is 52. Find the length and the width.

38. The width of a rectangle is 25% less than the length. The diagonal is 100. Find the area.

39. The length of a rectangle is 10. The diagonal is twice the shorter side. Find the width.

Fact from Geometry. A line drawn from one vertex of an equilateral triangle to the middle point of the opposite side is perpendicular to that side.



Then, in the equilateral triangle ABC, if D is the middle point of AC, BD is the altitude; and

$$\overline{BD}^2 = \overline{AB}^2 - \overline{AD}^2 = \overline{AB}^2 - \left(\frac{AC}{2}\right)^2.$$

40. If BC in the foregoing triangle is 6, find BD and the area of the triangle.

41. If AC in the foregoing triangle is 8, find BD and the area of the triangle.

42. If BD in the foregoing triangle is 14, find AB and the area of the triangle.

43. The perimeter of an equilateral triangle is 45. Find the altitude.

44. The altitude of an equilateral triangle is 25 centimeters. Find one side.

NOTE. A method of extracting the square root of numbers not unlike that in use to-day was employed by the Greek, Theon, about A.D. 350. In the Middle Ages square roots were extracted with a fair degree of accuracy by using the formulas of approximation:

$$(1)\ \sqrt{a^2+x}=a+\frac{x}{2a}. \qquad (2)\ \sqrt{a^2+x}=a+\frac{x}{2a+1}.$$

The true value of the square root of the number was proved to be between the results obtained by these expressions. Thus, if $\sqrt{65}$ was desired, it was noticed that $65 = 64 + 1$, and from (1)

$$\sqrt{65}=\sqrt{64+1}=\sqrt{8^2+1}=8+\frac{1}{2\cdot 8}=8\tfrac{1}{16},$$

while from (2)

$$\sqrt{65}=\sqrt{64+1}=\sqrt{8^2+1}=8+\frac{1}{2\cdot 8+1}=8\tfrac{1}{17}.$$

Thus the true value of $\sqrt{65}$ is between these two numbers. This method was known to the Arabs.

It should be kept in mind that the use of decimal fractions and of the decimal point was not common until the eighteenth century. Consequently the complete development of the method of extracting the square root given in the text is comparatively recent.

CHAPTER XX

RADICALS

103. Rational numbers. The quotient of two integers is called a **rational number.**

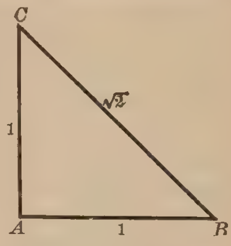
Any integer is a rational number, since it may be considered as the quotient of itself and 1.

Thus 5, -8, $\frac{2}{3}$, $-\frac{7}{8}$, and 4.693 are rational numbers.

104. Radical. A radical is an indicated root of an algebraic or arithmetical expression.

Thus $\sqrt{4}$, $\sqrt{3}$, $\sqrt[3]{a}$, and $\sqrt{x^2 - 5x + 6}$ are radicals.

If a number under a radical sign is such that the root cannot be taken exactly, the radical represents an **irrational number.**



Thus $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt{3}$, $\sqrt[3]{5}$, are irrational numbers, since the indicated roots of 2, of 3, and of 5 will never come out even however far the process of extracting the root is carried.

Though no irrational numbers can be expressed exactly in decimals, we can represent a few of them by the lengths of lines. Thus in the right triangle ABC, if $AB = AC = 1$ inch, $BC = \sqrt{2}$ inches. If AB were 2 inches and AC were 1 inch, BC would be $\sqrt{5}$ inches.

There are other types of irrational numbers which cannot be expressed in terms of radicals, but their consideration is too complicated for this text.

If a negative number occurs under a square-root sign, the radical represents an imaginary number.

Thus $\sqrt{-2}$, $\sqrt{-8}$, and $\sqrt{-4}$ are imaginary numbers.

If the student pursues the study of algebra he will learn that imaginary numbers are required to express completely the cube and higher roots of any positive or negative number.

For example, he will learn that the number 27 has *two other* cube roots besides the number 3.

105. Index. The small figure like the 4 in $\sqrt[4]{}$ is called the index of the radical.

The index determines the **order** of the radical and indicates the root to be extracted.

In $5\sqrt[3]{7}$, 3 is the index, and the radical is of the third order.

106. Radicand. The **radicand** is the number, or expression, under the radical sign.

In $\sqrt{7}$ and $\sqrt[3]{ax}$, 7 and ax are the radicands.

107. Principal root. For a given index the **principal root** of a number is its *one root*, if it has but one; or its *positive root*, if it has two roots.

The principal root of $\sqrt[3]{-27}$ is -3; that of $\sqrt[4]{16}$ is $+2$, not -2.

108. Fractional exponents. Radical expressions may be written in two ways, with *radical signs* or with *fractional exponents*. The relation between the two will now be explained. To do this it is necessary to extend the meaning of the term *exponent*, which as defined on page 9 applied to positive integral exponents only. We shall assume that the laws which govern the use of integral exponents hold for fractional exponents also.

The fact that $x^2 \cdot x^3 = x^5$ illustrates the more general law $x^a \cdot x^b = x^{a+b}$, where a and b represent either integers or fractions.

Accordingly $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1$ or x. Since $x^{\frac{1}{2}}$ multiplied by itself gives x, *$x^{\frac{1}{2}}$ must be another way of writing the square root of x.*

Hence $\sqrt{x}$ may be written $x^{\frac{1}{2}}$.

Then $4^{\frac{1}{2}} = \sqrt{4} = 2$, and $(25\,a^2)^{\frac{1}{2}} = \sqrt{25\,a^2} = 5\,a$.

Further, $x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x^1 = x$.

And since $x^{\frac{1}{3}}$ is one of the three equal numbers whose product is x, $x^{\frac{1}{3}}$ *is another way of writing the cube root of* x.

Therefore $\sqrt[3]{x}$ may be written $x^{\frac{1}{3}}$.

This means that $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$.

Similarly, $\sqrt[4]{x} = x^{\frac{1}{4}}$.

In general terms, $\boldsymbol{\sqrt[n]{x} = x^{\frac{1}{n}}}$.

Now $x^{\frac{3}{2}} = x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = (x^{\frac{1}{2}})^3 = (\sqrt{x})^3$,

and $x^{\frac{3}{2}} = x^{3 \cdot \frac{1}{2}} = (x^3)^{\frac{1}{2}} = \sqrt{x^3}$.

Hence $x^{\frac{3}{2}} = (\sqrt{x})^3$, or $\sqrt{x^3}$.

In general terms, $\boldsymbol{x^{\frac{a}{n}} = \sqrt[n]{x^a}}$.

Thus $x^{\frac{a}{n}}$ means the nth root of x to the ath power.

The student should fix in mind that the *denominator* of the fractional exponent is the *index* of the root, and the *numerator* the *power* to which the radicand is raised. Moreover, whether one extracts the root first and then raises the result to the power, or vice versa, depends wholly on convenience.

ORAL EXERCISES

Read in radical form:

1. $x^{\frac{2}{3}}$.	**6.** $5\,a^{\frac{3}{4}}$.	**11.** $3\,a^{\frac{5}{2}}(bc)^{\frac{1}{3}}$.
2. $x^{\frac{3}{2}}$.	**7.** $(5\,a)^{\frac{3}{4}}$.	**12.** $5^{\frac{1}{3}}x^{\frac{2}{3}}$.
3. $(cd)^{\frac{3}{2}}$.	**8.** $3\,rx^{\frac{2}{3}}$.	**13.** $4^{\frac{1}{3}}t^{\frac{1}{a}}$.
4. $(2\,x)^{\frac{1}{3}}$.	**9.** $h^{\frac{2}{3}}k^{\frac{2}{3}}$.	
5. $2\,x^{\frac{1}{3}}$.	**10.** $7\,s^{\frac{2}{3}}(t+w)^{\frac{1}{2}}$.	**14.** $2\,x^{\frac{1}{b}}y^{\frac{c}{d}}$.

Find the numerical values of:

15. $25^{\frac{1}{2}}$.

16. $27^{\frac{1}{3}}$.

17. $16^{\frac{1}{4}}$.

18. $4^{\frac{3}{2}}$.

19. $64^{\frac{2}{3}}$.

20. $125^{\frac{2}{3}}$.

21. $(-8)^{\frac{4}{3}}$.

22. $32^{\frac{1}{5}}$.

23. $81^{\frac{3}{4}}$.

24. $(-216)^{\frac{2}{3}}$.

25. $(\frac{1}{16})^{\frac{1}{2}}$.

26. $(\frac{1}{25})^{\frac{1}{2}}$.

27. $(\frac{1}{16})^{\frac{1}{4}}$.

28. $25^{\frac{1}{2}} \cdot 4^{\frac{5}{2}}$.

29. $4^{\frac{1}{2}} \cdot (\frac{1}{4})^{\frac{1}{2}}$.

30. What is the principal square root of 4? of 25? of 36? the principal cube root of $+8$? of -8? of -27? of $+27$? the principal fifth root of 32? of -32? of 243? of -243?

31. What is an index? a radicand? Illustrate.

EXERCISES

Write with fractional exponents and simplify results:

1. $\sqrt{a^8}$.

2. $\sqrt{ax^4}$.

3. $3\sqrt{2x^5}$.

4. $\sqrt{9x}$.

5. $5\sqrt{16ax^2}$.

6. $\sqrt[3]{a^2}$.

7. $\sqrt[3]{ax^4}$.

8. $2\sqrt[3]{2x^2}$.

9. $3\sqrt[3]{8x^4}$.

10. $4\sqrt[3]{27ax^3}$.

11. $2\sqrt[4]{a^2x^8}$.

12. $4\sqrt[4]{16x}$.

13. $12x^2\sqrt[3]{ax^3}$.

14. $c\sqrt{(de)^3}$.

15. $3\sqrt{a^3} \cdot \sqrt[3]{x^2}$.

16. $\sqrt[n]{x^a} \cdot \sqrt[n]{y^b}$.

17. $\sqrt[n]{x^u} \cdot \sqrt[n]{x^{3a}}$.

18. $(-32)^{\frac{4}{5}} \cdot \sqrt[3]{-64}$.

19. $36^{\frac{1}{2}} \cdot \sqrt{\frac{1}{9}}$.

20. $9^{\frac{1}{2}} \cdot \sqrt[3]{27}$.

21. $(\frac{1}{4})^{\frac{1}{2}} \cdot \sqrt[3]{8^2}$.

109. Simplification of radicals. The form of a radical expression may be changed without altering its numerical value. Such changes are necessary for many reasons. For example, the numerical value of a radical expression is most easily obtained from its simplest form. It will be made clear later that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Granting that the two fractions are really equal, one can see by inspection that the value to several decimals can be computed more easily from the second fraction than from the first.

EXAMPLES

Study the following changes of form:

1. $\sqrt{36} = \sqrt{4 \cdot 9} = \sqrt{4} \cdot \sqrt{9} = 2 \cdot 3 = 6.$

2. Similarly, $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}.$

3. More generally, $\sqrt{a^2b} = \sqrt{a^2}\sqrt{ab} = a\sqrt{ab}.$

4. Also $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3} = \sqrt[3]{8}\sqrt[3]{3} = 2\sqrt[3]{3}.$

5. More generally, $\sqrt[3]{a^3b} = \sqrt[3]{a^3}\sqrt[3]{b} = a\sqrt[3]{b}.$

6. Finally, $\sqrt[n]{a^nb} = \sqrt[n]{a^n}\sqrt[n]{b} = a\sqrt[n]{b}.$

The six preceding examples illustrate the correctness of the following rule for simplifying a radical involving the square root of an integer or an integral expression.

Rule. *Separate the radicand into two factors one of which is the greatest perfect square which it contains. Then take the square root of this factor and write it as the coefficient of a radical of which the other factor is the radicand.*

If the radical already has a coefficient other than the number 1, multiply the result obtained above by this coefficient.

A similar rule holds for radicals involving the cube and higher roots.

Thus $\sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = 2\sqrt[3]{2},$

and $\sqrt[5]{96} = \sqrt[5]{32 \cdot 3} = 2\sqrt[5]{3}.$

NOTE. Although the Arabs were by no means able to state all the rules explained in this chapter, it is interesting to note that they did recognize the truth of a few of them. For instance, a writer about A.D. 830 gives, in his own notation, of course, the facts contained in the formulas $\sqrt{a^2b} = a\sqrt{b}$, and $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

EXERCISES

Simplify:

1. $\sqrt{8}$.
2. $\sqrt{12}$.
3. $\sqrt{18}$.
4. $\sqrt{20}$.
5. $\sqrt{45}$.
6. $\sqrt{50}$.
7. $\sqrt{75}$.
8. $\sqrt{63}$.
9. $\sqrt{98}$.
10. $2\sqrt{72}$.
11. $3\sqrt{80}$.
12. $5\sqrt{128}$.
13. $\sqrt{147}$.
14. $3\sqrt{162}$.
15. $10\sqrt{175}$.
16. $\sqrt{192}$.
17. $\sqrt{243}$.
18. $\sqrt{245}$.
19. $\sqrt{343}$.
20. $\sqrt{363}$.
21. $2\sqrt{720}$.
22. $\sqrt{1250}$.
23. $\sqrt[3]{16}$.
24. $\sqrt[3]{24}$.
25. $2\sqrt[3]{40}$.
26. $3\sqrt[3]{56}$.
27. $4\sqrt[3]{72}$.
28. $5\sqrt[3]{96}$.
29. $\sqrt[3]{54}$.
30. $\sqrt[3]{81}$.
31. $\sqrt[3]{135}$.
32. $3\sqrt[3]{189}$.
33. $7\sqrt[3]{128}$.
34. $\sqrt[3]{192}$.
35. $\sqrt[3]{250}$.
36. $8\sqrt[3]{375}$.
37. $\sqrt[3]{448}$.
38. $2\sqrt[3]{625}$.
39. $\sqrt{a^3}$.

 HINT. $\sqrt{a^3} = \sqrt{a^2 \cdot a}$.

40. $\sqrt{a^5}$.
41. $\sqrt{4x^3}$.
42. $\sqrt{8x^2}$.
43. $\sqrt{8x^3}$.
44. $\sqrt{a^3x^2}$.
45. $2\sqrt{x^3a}$.
46. $3\sqrt{8a^3x}$.
47. $5x\sqrt{4a^5x^3}$.
48. $\sqrt[3]{a^4}$.

 HINT. $\sqrt[3]{a^4} = \sqrt[3]{a^3 \cdot a}$.

49. $\sqrt[3]{a^5}$.
50. $2a\sqrt[3]{x^7}$.
51. $\sqrt[3]{8x^4}$.
52. $\sqrt[3]{27x^5}$.
53. $3x\sqrt[3]{4a^3x}$.
54. $5x\sqrt[3]{4a^2x^3}$.
55. $\sqrt[3]{27a^3x}$.
56. $\sqrt[3]{56a^4x}$.
57. $\sqrt{8+4\sqrt{2}}$.

 Solution. $\sqrt{8+4\sqrt{2}} = \sqrt{4(2+\sqrt{2})} = 2\sqrt{2+\sqrt{2}}$.

58. $\sqrt{4-4\sqrt{3}}$.
59. $\sqrt{18+9\sqrt{3}}$.
60. $\sqrt{100-25\sqrt{5}}$.
61. $\sqrt{9\sqrt{2}-27}$.
62. $\sqrt{R^2-R^2\sqrt{3}}$.
63. $\sqrt{R^2+4R^2\sqrt{5}}$.
64. $\sqrt{5^2\sqrt{3}-25^2}$.
65. $\sqrt{a^2x^2+a^3\sqrt{x}}$.

The foregoing exercises are easier to simplify than radicals whose radicands are fractions or fractional expressions. The latter arise frequently in practice, especially in certain parts of geometry.

EXAMPLES

Study the following simplification of fractional radicands:

1. $\sqrt{\frac{3}{2}} = \sqrt{\frac{6}{4}} = \sqrt{\frac{1}{4} \cdot 6} = \sqrt{\frac{1}{4}}\sqrt{6} = \frac{1}{2}\sqrt{6}.$

2. $6\sqrt{\frac{1}{3}} = 6\sqrt{\frac{3}{9}} = 6\sqrt{\frac{1}{9} \cdot 3} = 6 \cdot \frac{1}{3}\sqrt{3} = 2\sqrt{3}.$

3. $\sqrt{\frac{3}{5x}} = \sqrt{\frac{15x}{25x^2}} = \sqrt{\frac{1}{25x^2} \cdot 15x} = \frac{1}{5x}\sqrt{15x}.$

These examples illustrate the following rule for simplifying a square root which has a *fractional* radicand.

Rule. *Multiply the numerator and the denominator of the radicand by the least whole number or simplest expression which will make the resulting denominator a perfect square.*

Then separate the radicand into two factors, one of which is a fraction and at the same time the greatest perfect square which the radicand contains.

Take the square root of this factor and write it as the coefficient of the radical of which the other factor is the radicand. If the original radical has a coefficient, multiply the result as obtained above by this coefficient.

A similar rule holds for simplifying a cube root and radicals of higher orders which have fractional radicands.

EXERCISES

Simplify the following:

1. $\sqrt{\frac{3}{4}}$.	6. $\sqrt{\frac{3}{2}}$.	11. $\sqrt{\frac{5}{8}}$.	16. $\sqrt{\frac{5}{12}}$.
2. $\sqrt{\frac{5}{9}}$.	7. $\sqrt{\frac{3}{5}}$.	12. $\sqrt{\frac{5}{2}}$.	17. $3\sqrt{\frac{7}{12}}$.
3. $\sqrt{\frac{3}{16}}$.	8. $\sqrt{\frac{5}{3}}$.	13. $3\sqrt{\frac{3}{7}}$.	18. $2\sqrt{\frac{5}{18}}$.
4. $\sqrt{\frac{1}{2}}$.	9. $\sqrt{\frac{4}{7}}$.	14. $2\sqrt{\frac{2}{5}}$.	19. $3a\sqrt{\frac{3}{11}}$.
5. $\sqrt{\frac{2}{3}}$.	10. $\sqrt{\frac{7}{5}}$.	15. $\sqrt{\frac{8}{5}}$.	20. $3a^2\sqrt{\frac{3}{32}}$.

21. $\sqrt{1-(\frac{1}{2})^2}$.

HINT. $\sqrt{1-(\frac{1}{2})^2}=\sqrt{1-\frac{1}{4}}=\sqrt{\frac{3}{4}}$, etc.

22. $\sqrt{1-(\frac{1}{3})^2}$.

23. $\sqrt{9-(\frac{3}{2})^2}$.

24. $\sqrt{25-(\frac{5}{2})^2}$.

25. $\sqrt{2+(\frac{3}{8})^2}$.

26. $\sqrt{81-(\frac{9}{2})^2}$.

27. $\sqrt{121-(\frac{11}{2})^2}$.

28. $\sqrt{169-(\frac{13}{2})^2}$.

29. $\sqrt{\frac{R}{3}}$.

30. $\sqrt{\frac{R^2}{3}}$.

31. $\sqrt{\frac{3x^2}{4}}$.

32. $\sqrt{R^2+R^2}$.

33. $\sqrt{R^2-\left(\frac{R}{2}\right)^2}$.

34. $\sqrt{R^2-\left(\frac{R}{4}\right)^2}$.

35. $\sqrt{\frac{3R^2}{4}-\frac{R^2}{4}\sqrt{2}}$.

36. $\sqrt{\frac{R^2}{4}+\frac{R^2x}{2}}$.

37. $\sqrt{\frac{R^2}{4}+\frac{R^2}{2}\sqrt{2}}$.

38. $\sqrt{R^2\sqrt{2}-\frac{R^2}{2}}$.

39. $\sqrt[3]{R^3-\left(\frac{R}{2}\right)^3}$.

40. $\sqrt[3]{16+8\sqrt{2}}$.

41. $\sqrt[3]{81+3\sqrt{243}}$.

The need for simplifying radicals presents itself in various problems, as, for example, in simple geometrical work on right triangles.

PROBLEMS

(Obtain answers in *simplest radical form.*)

1. One leg of a right triangle is 8 and the other is 10. Find the hypotenuse.

Solution. $x=\sqrt{8^2+10^2}=\sqrt{164}=\sqrt{4\cdot 41}=2\sqrt{41}$.

2. The hypotenuse of a right triangle is 8 and one leg is 4. Find the other leg and the area.

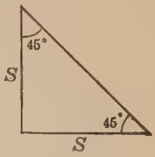
3. The hypotenuse of a right triangle is R and one leg is $\frac{R}{2}$. Find the other leg and the area.

4. Find the diagonal of a square whose side is 12.

5. Find the sides of a square whose diagonal is 12.

6. Find the sides of a square whose diagonal is $2R$.

Problems involving the following classes of triangles are of frequent occurrence in practical work and often require the use of radicals:

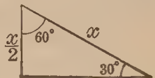


(*a*) An isosceles right triangle; that is, a right triangle with two equal sides.

As indicated in the figure, if each leg is S the two acute angles are 45° each.

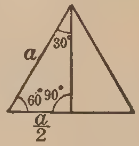
(*b*) A right triangle with one angle 30° or 60°.

As indicated in the figure, if one acute angle is 30° the other is 60°, and vice versa. More important still, the *hypotenuse* is always *twice the shorter leg*.

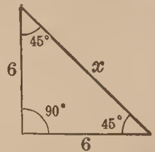


(*c*) An equilateral triangle.

As indicated in the figure, when the altitude is drawn, it divides the base into two equal parts, and in each of the right triangles formed the same relations exist as in (*b*), above.



7. If each leg of an isosceles right triangle is 6, find the hypotenuse.



Hint. $x^2 = 6^2 + 6^2$.

$x = \sqrt{36 + 36}$, etc.

8. Find the hypotenuse of an isosceles right triangle if one leg is 8; if one leg is 13.

9. The hypotenuse of an isosceles right triangle is 10. Find the legs.

Hint. $x^2 + x^2 = 100$. $x^2 = 50$, etc.

10. The hypotenuse of an isosceles right triangle is 13. Find the legs.

11. One angle of a right triangle is 30°. The hypotenuse is 20. Find the other two sides.

Hint. See (*b*), above.

12. One angle of a right triangle is 60°. The hypotenuse is 12. Find the other two sides.

13. One angle of a right triangle is 30°. The leg opposite it is 10. Find the hypotenuse and the other leg.

14. One angle of a right triangle is 60° and the adjacent leg is 12. Find the other two sides.

15. The side of an equilateral triangle is 12. Find the altitude and the area.

HINT. See (*c*) and (*b*), page 258.

16. The side of an equilateral triangle is S. Find the altitude and the area.

17. The altitude of an equilateral triangle is 10. Find the side and the area.

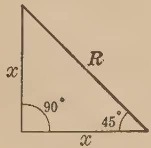
18. The legs of a right triangle are equal. Its hypotenuse is 20. Find the legs of the triangle.

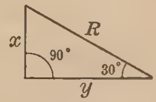
19. The legs of a right triangle are equal and its area is 50. Find the hypotenuse.

20. The legs of a right triangle are $\frac{R}{2}$ and $\frac{3R}{2}$. Find the hypotenuse.

21. One leg of a right triangle is $\frac{R}{5}$. The hypotenuse is R. Find the other leg.

Find x and y in terms of R in the following:

22. 

23. 

24. The legs of a right triangle are R and $\frac{R}{2}(a-1)$. Find the hypotenuse.

110. Addition and subtraction of radicals. Radicals of the same order which are in their simplest form and have like radicands are really similar terms. They can be added

or subtracted and the result expressed by one term according to the rule on page 33.

Thus $3\sqrt{2}-7\sqrt{2}+9\sqrt{2}=5\sqrt{2}$.

Similarly, $7\sqrt{\frac{1}{2}}-\frac{3}{2}\sqrt{2}=\frac{7}{2}\sqrt{2}-\frac{3}{2}\sqrt{2}=2\sqrt{2}$.

The last example illustrates the necessity of acquiring the ability to simplify radicals before attempting to carry out the fundamental operations of addition and subtraction.

If the radicands are unlike and cannot be simplified further, the radicals are really dissimilar terms, and addition or subtraction can only be indicated. (See page 34.)

Thus $\sqrt{2}$, $\sqrt{3}$, and $\sqrt[3]{5}$ are three dissimilar radicals, and the addition of the three can only be indicated thus: $\sqrt{2}+\sqrt{3}+\sqrt[3]{5}$.

EXERCISES

Simplify and collect:

1. $\sqrt{27}+\sqrt{12}$.
2. $\sqrt{45}-\sqrt{20}$.
3. $2\sqrt{200}-3\sqrt{8}$.
4. $\sqrt{28}+\sqrt{63}$.
5. $7\sqrt{18}-\sqrt{98}$.
6. $\sqrt{75}-\sqrt{27}+2\sqrt{48}$.
7. $\sqrt{2}+\sqrt{\frac{1}{2}}$.
8. $\sqrt{3}-\sqrt{\frac{1}{3}}$.
9. $\frac{3}{2}+\sqrt{\frac{3}{4}}$.
10. $\frac{2}{3}+\sqrt{\frac{4}{3}}$.
11. $\frac{R}{2}+\sqrt{\frac{9R^2}{2}}$.
12. $\frac{3R}{2}-\sqrt{\frac{9R^2}{2}}$.
13. $R-\sqrt{\frac{3R^2}{4}}$.
14. $5\sqrt{\frac{1}{2}}-\frac{3}{2}\sqrt{2}$.
15. $8\sqrt{\frac{9}{2}}-\sqrt{72}$.
16. $\sqrt{\frac{1}{3}}+2\sqrt{\frac{4}{3}}-3\sqrt{\frac{2}{3}}$.
17. $\sqrt{\frac{2}{5}}-\sqrt{\frac{9}{10}}+2\sqrt{10}$.
18. $\sqrt{\frac{3}{10}}-\sqrt{120}-2\sqrt{\frac{6}{5}}$.
19. $\sqrt{\frac{5}{18}}+2\sqrt{\frac{32}{5}}-\sqrt{\frac{10}{9}}$.
20. $\sqrt[3]{56}+2\sqrt[3]{189}$.
21. $2\sqrt[3]{320}-\sqrt{50}$.
22. $\sqrt[4]{25}-\sqrt{20}$.

HINT. $\sqrt[4]{25}=\sqrt[4]{5^2}=5^{\frac{2}{4}}=5^{\frac{1}{2}}=\sqrt{5}$, etc.

23. $\sqrt[4]{9}+\sqrt{12}$.

24. $\sqrt[4]{32}+5\sqrt[4]{162}$.

25. $2x\sqrt[3]{54x}-3\sqrt[3]{16x^4}+\sqrt[6]{4x^2}$.

26. $\sqrt[3]{81x^7}+x\sqrt[3]{375x^4}-\sqrt[12]{16x^4}$.

27. $\sqrt{a^3bc}-a\sqrt{abc}+ac\sqrt{\frac{b}{ac}}$.

28. $rs\sqrt[3]{rs}+\sqrt[3]{\frac{1}{r^2s^2}}-2\sqrt[3]{r^4s^4}$.

29. $\sqrt{a^3+4a^2+4a}-\sqrt{a^3}-\frac{2}{a^3}\sqrt{a^7}$.

30. $\sqrt{3x^2-18x+27}-\sqrt{27(x^2+2x+1)}$.

NOTE. Though methods of classifying irrational expressions are found in the work of Euclid, the Hindus and the Arabs were the first to develop this part of algebra in anything like the form used to-day.

111. Multiplication of radicals. Monomial radicals of the same order are multiplied as follows:

EXAMPLES

1. Multiply $3\sqrt{8}$ by $2\sqrt{5}$.

Solution. $3\sqrt{8}\cdot 2\sqrt{5}=6\sqrt{40}=6\cdot 2\sqrt{10}=12\sqrt{10}$.

2. Multiply $5\sqrt[3]{4ax^2}$ by $\sqrt[3]{2a^2x^2}$.

Solution. $5\sqrt[3]{4ax^2}\cdot\sqrt[3]{2a^2x^2}=5\sqrt[3]{8a^3x^4}=10ax\sqrt[3]{x}$.

The method just illustrated of multiplying monomial radicals of the same order may be stated in the

Rule. *Take the product of the coefficients of the radicals for the coefficient of the radical in the result.*

Multiply together the radicands and write the product under the common radical sign.

Reduce the result to its simplest form.

The preceding rule does not hold for radicals of different orders.

EXERCISES

Perform the indicated operation and simplify results:

1. $\sqrt{5}\cdot\sqrt{6}$.
2. $\sqrt{7}\cdot\sqrt{7}$.
3. $2\sqrt{3}\cdot 3\sqrt{2}$.
4. $3\sqrt{2}\cdot 5\sqrt{2}$.
5. $\sqrt{2}\cdot\sqrt{8}$.
6. $\sqrt{3}\cdot\sqrt{27}$.
7. $5^{\frac{1}{2}}\cdot 20^{\frac{1}{2}}$.
8. $18^{\frac{1}{2}}\cdot 8^{\frac{1}{2}}$.
9. $\sqrt{\frac{12}{25}}\cdot\sqrt{75}$.
10. $\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{27}{8}}$.
11. $\sqrt{11}\cdot\sqrt{\frac{1}{11}}$.
12. $a^{\frac{1}{2}}\cdot(bc)^{\frac{1}{2}}$.
13. $R\sqrt{2}\cdot R\sqrt{3}$.
14. $\left(\sqrt{\frac{R}{2}}-\sqrt{2}\right)^2$.
15. $\frac{R}{2}\sqrt{2}\cdot R\sqrt{3}$.
16. $2\sqrt{x}\cdot\sqrt{4x^3}$.
17. $2\sqrt{rs}\cdot 7\sqrt{r^3s^3t^2}$.
18. $\sqrt{75a}\cdot(45a)^{\frac{1}{2}}$.
19. $\sqrt{2u}\cdot\sqrt{4v}\cdot\sqrt{6uv}$.
20. $5\sqrt{3m}\cdot 5\sqrt{3m}$.
21. $(3\sqrt{3x})^2$.
22. $\sqrt[3]{16}\cdot\sqrt[3]{4}$.
23. $\sqrt[3]{4}\cdot\sqrt[3]{12}$.
24. $(100)^{\frac{1}{3}}\cdot(30)^{\frac{1}{3}}$.
25. $\sqrt[4]{8}\cdot\sqrt[4]{32}$.
26. $5\sqrt[3]{2a}\cdot 3\sqrt[3]{16a}$.
27. $(\sqrt{R}-\sqrt{2})^2$.
28. $\left(\frac{R}{2}\sqrt{2}-\sqrt{5}\right)^2$.
29. $\sqrt{\frac{a}{x}}\cdot\sqrt{\frac{4x}{a}}$.
30. $\sqrt{\frac{5x}{a}}\cdot\sqrt{\frac{a^3}{5x^2}}$.

Find the radical expression having the coefficient 1 equivalent to each of the following:

31. $5\sqrt{2}$.

Solution. $5\sqrt{2}=\sqrt{25}\sqrt{2}=\sqrt{50}$.

32. $7\sqrt{3}$.
33. $10\sqrt{5}$.
34. $8\sqrt{\frac{1}{2}}$.
35. $12\sqrt{\frac{5}{3}}$.
36. $a\sqrt{x}$.
37. $2a\sqrt{c}$.
38. $3\sqrt[3]{2}$.

HINT. $3\sqrt[3]{2}=\sqrt[3]{27}\cdot\sqrt[3]{2}$.

39. $5\sqrt[3]{2}$.
40. $2\sqrt[3]{3}$.
41. $2\sqrt[3]{4}$.
42. $3\sqrt[3]{\frac{1}{3}}$.
43. $a\sqrt[3]{\frac{1}{a}}$.
44. $(a+x)\sqrt[3]{\frac{1}{a+x}}$.

The multiplication of binomial or of polynomial radical expressions of the same order involves no new principle.

EXAMPLE

Multiply $3\sqrt{5}-4\sqrt{3}$ by $2\sqrt{5}+\sqrt{3}$.

Solution.

$$\begin{array}{l} 3\sqrt{5}-4\sqrt{3} \\ \underline{2\sqrt{5}+\ \ \sqrt{3}} \\ 30\quad -8\sqrt{15} \\ \underline{\qquad +3\sqrt{15}-12} \\ 30\quad -5\sqrt{15}-12=18-5\sqrt{15}. \end{array}$$

EXERCISES

Perform the indicated multiplication and simplify the results:

1. $(\sqrt{2}-3)(3\sqrt{2}+5)$.
2. $(2\sqrt{5}-4)(3\sqrt{5}+3)$.
3. $(4\sqrt{7}-3)^2$.
4. $(2\sqrt{7}-3\sqrt{2})^2$.
5. $(3\sqrt{2}+\sqrt{3})(2\sqrt{3}-\sqrt{2})$.
6. $(2\sqrt{5}-3\sqrt{2})(3\sqrt{5}+2\sqrt{2})$.
7. $(\sqrt{5}-\sqrt{3}+\sqrt{2})(\sqrt{5}-\sqrt{3}+\sqrt{2})$.
8. $(3\sqrt{2}+2\sqrt{3}+\sqrt{30})(\sqrt{2}+\sqrt{3}-\sqrt{5})$.
9. $(R\sqrt{2}-2)(R\sqrt{2}-3)$.
10. $(R\sqrt{3}-R\sqrt{2})(R\sqrt{3}+2R\sqrt{2})$.
11. $\left(R-\frac{R}{2}\sqrt{2}\right)\left(R-\frac{R}{2}\sqrt{2}\right)$.
12. $\left(\frac{R}{2}+\frac{R}{2}\sqrt{3}\right)^2$.
13. $\frac{R}{2}(\sqrt{5}-1)^2$.
14. $R^2-\left(\frac{R}{2}\sqrt{5}-\frac{R}{2}\right)^2$.
15. $R^2-\left(\frac{R}{2}-\frac{R}{2}\sqrt{2}\right)^2$.
16. $\sqrt{R^2-\left(\frac{R}{2}-\frac{R}{2}\sqrt{3}\right)^2}$.
17. $\sqrt{R^2-\left(\frac{R\sqrt{5}-R}{4}\right)^2}$.

18. The hypotenuse of a right triangle is R and one leg is $R-\frac{R}{2}\sqrt{3}$. Find the other leg.

19. One leg of a right triangle is $\frac{R}{2}(\sqrt{5}-1)$ and the hypotenuse is R. Find the other leg.

20. The legs of a right triangle are $R\sqrt{2+\sqrt{2}}$ and $\frac{R}{2}\sqrt{2-\sqrt{2}}$. Find the hypotenuse and the area.

21. Show by substituting and simplifying that $2+2\sqrt{3}$ is a root of the quadratic equation $x^2-4x=8$.

22. Does $\frac{7}{2}-\frac{1}{2}\sqrt{41}$ satisfy $x^2-7x+2=0$?

112. Radical equations. An irrational or radical equation in one unknown is an equation in which the unknown occurs in a radicand. The solution of the simpler types of radical equations depends mainly on a knowledge of multiplication of radicals. The ability to square or to cube each member of an equation and the exercise of a little judgment *in transposing at the proper time* are sufficient to solve any of the equations which arise in elementary work.

The necessity of checking all results obtained cannot be too strongly emphasized. The pupil should remember that an answer is a *root* on the one condition that it *satisfies* the *original equation.*

EXERCISES

Solve for x and check:

1. $\sqrt{2x-7}=3$.

HINT. Squaring each member, $2x-7=9$, etc.

2. $\sqrt{3x-5}=5$.

3. $\sqrt{x-6}=\sqrt{5}$.

4. $\sqrt{2x+3}=\sqrt{7}$.

5. $2\sqrt{x-1}=\sqrt{6}$.

6. $\sqrt{x-3}-6=0$.

HINT. Transpose before squaring.

7. $2\sqrt{3x+1}-8=0$.

8. $2\sqrt{5x-4}+3=11$.

9. $\sqrt[3]{x-1}=2$.

10. $\sqrt[3]{2x+1}=3$.

11. $\sqrt[3]{4x-6}=4.$

12. $\sqrt[3]{2x-1}+5=0.$

13. $3\sqrt[3]{2x+7}=6.$

14. $2\sqrt[3]{3x-9}+4=0.$

15. $3\sqrt{x-2}=\sqrt{2x+3}.$

16. $2\sqrt{x+3}=\sqrt{x+18}.$

17. $3\sqrt{2x-6}=2\sqrt{x+4}.$

18. $\sqrt{2x-1}-\sqrt{2-x}=0.$

HINT. Transpose, then square.

19. $\sqrt{4x-3}-\sqrt{3x+5}=0.$

20. $\sqrt{6x-5}-\sqrt{5x+7}=0.$

21. $2\sqrt{3x-3}-3\sqrt{2+x}=0.$

22. $3-\sqrt{x+3}=\sqrt{x}.$

HINT. Squaring each member, $9-6\sqrt{x+3}+x+3=x.$
Transposing, $-6\sqrt{x+3}=-12.$
Dividing by -6, $\sqrt{x+3}=2.$
Squaring again, $x+3=4$, etc.

23. $5-\sqrt{2x+5}=\sqrt{2x}.$

24. $7-\sqrt{3x+7}-\sqrt{3x}=0.$

HINT. $7-\sqrt{3x}=\sqrt{3x+7}$, etc.

25. $\sqrt{3x-5}+\sqrt{3x+7}=6.$

26. $\sqrt{x}+\sqrt{2}-\sqrt{x+2}=0.$

27. $\sqrt{x+1}-\sqrt{2x-3}=\sqrt{3x-2}.$

HINT. Squaring, $x+1-2\sqrt{(x+1)(2x-3)}+2x-3=3x-2.$
Transposing, $2\sqrt{(x+1)(2x-3)}=0$, etc.

28. $\sqrt{x-3}+\sqrt{2x+4}=\sqrt{3x+1}.$

29. $\sqrt{x+3}+\sqrt{x-4}=\sqrt{4x-3}.$

30. $\sqrt{x+4}+\sqrt{x-3}=\sqrt{4x-3}.$

31. $\dfrac{\sqrt{x}-3}{\sqrt{x}+1}=\dfrac{\sqrt{x}-4}{\sqrt{x}-2}.$

32. $\dfrac{\sqrt{x}-3}{\sqrt{x}+2}=\dfrac{\sqrt{x}-5}{\sqrt{x}-1}.$

113. Division of radicals. It is frequently necessary to find the approximate value of an expression which involves division by a radical expression. Thus $2 \div \sqrt{3}$, $\sqrt{3} \div \sqrt{5}$, $\frac{4-\sqrt{3}}{2-\sqrt{3}}$, and $\frac{3\sqrt{2}}{\sqrt{5}+\sqrt{2}}$ are types which often occur.

To find the approximate value of $2 \div \sqrt{3}$ we may extract the square root of 3 to several decimal places and then divide 2 by the approximate root obtained. Both of these processes are long and one of them is unnecessary. For, writing $2 \div \sqrt{3}$ as a fraction and multiplying both terms by $\sqrt{3}$ gives $\frac{2\sqrt{3}}{3}$. To find the approximate value of this last fraction requires but one long operation.

Similarly, the process of finding the approximate value of $\sqrt{7} \div (\sqrt{7} - \sqrt{2})$ involves three rather lengthy operations — the extracting of two square roots, and one long division. The labor of two of these operations can be avoided in a manner similar to that shown above.

114. Rationalizing factors. One radical expression is the **rationalizing factor** for another if the *product* of the two is *rational.*

A rationalizing factor of $\sqrt{3}$ is $\sqrt{3}$, for $\sqrt{3} \cdot \sqrt{3} = 3$.

For $\sqrt[3]{2}$ a rationalizing factor is $\sqrt[3]{4}$, since $\sqrt[3]{2} \cdot \sqrt[3]{4} = \sqrt[3]{8} = 2$.

Similarly, $\sqrt{7} - \sqrt{2}$ is a rationalizing factor of $\sqrt{7} + \sqrt{2}$, as their product $(\sqrt{7} - \sqrt{2})(\sqrt{7} + \sqrt{2}) = 7 - 2 = 5$.

In like manner $(3\sqrt{5} - 2\sqrt{3})(3\sqrt{5} + 2\sqrt{3}) = 45 - 12 = 33$. Therefore $3\sqrt{5} - 2\sqrt{3}$ is a rationalizing factor of $3\sqrt{5} + 2\sqrt{3}$.

The binomial radicals of the last two illustrations are of the general types $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$. Such binomials are called **conjugate** radicals and *either* is a rationalizing factor of the other. If a and b are rational, the product $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$, or $a - b$, is also *rational.*

ORAL EXERCISES

Determine a rationalizing factor for each of the following expressions and find the product of the given expression and this factor:

1. $\sqrt{5}$.	6. $\sqrt{27}$.	11. $\sqrt[3]{25}$.	16. $4\sqrt{3}-\sqrt{2}$.
2. $3\sqrt{6}$.	7. $\sqrt[3]{2}$.	12. $\sqrt{2}+3$.	17. $2\sqrt{5}+7\sqrt{6}$.
3. $2\sqrt{7}$.	8. $\sqrt[3]{3}$.	13. $\sqrt{3}-\sqrt{2}$.	18. $\sqrt{x}-\sqrt{a}$.
4. $\sqrt{8}$.	9. $2\sqrt[3]{5}$.	14. $3+\sqrt{7}$.	19. $\sqrt{3a}+\sqrt{x}$.
5. $\sqrt{32}$.	10. $\sqrt[3]{16}$.	15. $3\sqrt{2}-5$.	20. $\sqrt[3]{a-2x}$.

Direct division of similar radicals, coefficient by coefficient and radicand by radicand, is often possible.

Thus $$\sqrt{6}\div\sqrt{2}=\sqrt{3}.$$

And $$12\sqrt{10}\div 2\sqrt{5}=6\sqrt{2}.$$

But $$\sqrt{7}\div\sqrt{3}=\frac{\sqrt{7}}{\sqrt{3}}=\frac{\sqrt{7}\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}=\frac{\sqrt{21}}{3}.$$

If direct division cannot be exactly performed, we use the

Rule. *Write the dividend over the divisor in the form of a fraction. Then multiply the numerator and denominator of the fraction by the rationalizing factor of the denominator and simplify the resulting fraction.*

EXERCISES

Perform the indicated division:

1. $\sqrt{10}\div\sqrt{2}$.
2. $\sqrt{18}\div\sqrt{3}$.
3. $\sqrt{5}\div\sqrt{2}$.

Solution. $\sqrt{5}\div\sqrt{2}$.

$$\frac{\sqrt{5}}{\sqrt{2}}=\frac{\sqrt{5}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}=\frac{\sqrt{10}}{2}.$$

4. $\sqrt{7}\div\sqrt{3}$.
5. $2\sqrt{6}\div 3\sqrt{2}$.
6. $\sqrt{10}\div\sqrt{3}$.
7. $2\sqrt{7}\div\sqrt{2}$.
8. $\sqrt{6}\div\sqrt{18}$.

9. $6 \div 2\sqrt{2}$.

10. $8\sqrt{15} \div 4\sqrt{5}$.

11. $3\sqrt{2} \div 15\sqrt{8}$.

12. $\sqrt{6} - \sqrt{18} \div \sqrt{2}$.

13. $(\sqrt{12} - \sqrt{24}) \div 3\sqrt{3}$.

14. $(\sqrt{6} - \sqrt{9} + 18) \div 2\sqrt{2}$.

15. $8 \div (3 + \sqrt{7})$.

Solution. $$8 \div (3 + \sqrt{7}) = \frac{8}{3 + \sqrt{7}} = \frac{8(3 - \sqrt{7})}{(3 + \sqrt{7})(3 - \sqrt{7})}$$
$$= \frac{24 - 8\sqrt{7}}{9 - 7} = 12 - 4\sqrt{7}.$$

16. $9 \div (3 - \sqrt{2})$.

17. $\sqrt{5} \div (\sqrt{5} - \sqrt{2})$.

HINT. $$\frac{\sqrt{5}}{\sqrt{5} - \sqrt{2}} = \frac{\sqrt{5}(\sqrt{5} + \sqrt{2})}{(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})}, \text{ etc.}$$

18. $\dfrac{2\sqrt{3}}{\sqrt{3} + \sqrt{2}}$.

19. $\dfrac{4\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}}$.

20. $\dfrac{\sqrt{3} + \sqrt{7}}{\sqrt{7} - 2}$.

Find to three decimals the value of:

21. $5 + 3\sqrt{2}$.

22. $7 - 3\sqrt{5}$.

23. $\dfrac{7 + \sqrt{6}}{3}$.

24. $\dfrac{\sqrt{2}}{\sqrt{3}}$.

25. $\dfrac{\sqrt{5} + 1}{\sqrt{5}}$.

26. $\dfrac{2}{\sqrt{3} - \sqrt{2}}$.

27. $3\sqrt{8} - \sqrt{7}$.

28. $\sqrt{3 - \sqrt{2}}$.

Simplify:

29. $\sqrt{a} \div \sqrt{x}$.

30. $\sqrt{3a^2} \div \sqrt{ax^2}$.

31. $2\sqrt{ax} \div 3\sqrt{bx}$.

BIOGRAPHICAL NOTE. *François Vieta.* The reason that algebra is a universal language which does not depend entirely on the nationality of the writer lies in the fact that the symbols used to indicate the various operations and relations are widely understood and adopted. This has not always been the case, and for a long time during the early history of the subject there was no accepted notation in algebra, but each man used any symbol that suited him. One of the men who did most to establish a fixed notation was François Vieta (1540–1603), a French lawyer who studied and wrote on mathematics as a pastime. He was in public life during his whole career and was well known for his ability to decipher the hidden meaning of dispatches captured from the enemy.

FRANÇOIS VIETA.

It was he who established the use of the signs + and − for addition and subtraction, which, to be sure, had been used before his time, but were not generally accepted. He also denoted the known numbers in an equation by the consonants, B, C, D, etc., and the unknowns by the vowels, A, E, I, etc. He recognized the existence of negative roots of equations, but rejected them as absurd.

To denote the second and third powers of the unknown, he used the letters Q (*quadratus*) and C (*cubus*) respectively. Instead of using the sign = he wrote aeq. (*aequalis* or *aequatur*). Thus Vieta would have written the equation $x^3 - 8x^2 + 16x = 40$ in the form

$$1\,C - 8\,Q + 16\,N \text{ aeq. } 40.$$

Before the time of Vieta this equation would have been written in a much more primitive notation. For instance, with writers only a little earlier it would appear as

$$\text{Cubus } \bar{m} \text{ 8 Census } \bar{p} \text{ 16 rebus aequatur 40.}$$

It is easily seen that operations on equations in this form would be very hard to perform.

Vieta is further distinguished as being the first man to obtain an exact numerical expression for the number π, which occurs in geometry. His form of expression calls for an infinite number of operations which, of course, could never be performed, but the further one proceeded, the closer would be the approximation obtained. In a certain sense the familiar sign $\sqrt{\ }$ implies an infinite number of operations, for one can never go through the process of extracting the square root of 2, for instance, and come out even. Vieta's method of denoting π was, however, more involved than this and made use of complicated irrational fractions.

CHAPTER XXI

QUADRATIC EQUATIONS

115. Solution by completing the square. The quadratic equation was defined on page 137. Before taking up the work that follows, the student should review the method of forming trinomial squares given on page 121.

ORAL EXERCISES

What terms should be added in order to make the following expressions perfect squares?

1. $x^2+2x+?$	**5.** $x^2+18x+?$	**9.** $x^2-5x+?$
2. $x^2+4x+?$	**6.** $x^2-18x+?$	**10.** $x^2+7x+?$
3. $x^2-10x+?$	**7.** $x^2+3x+?$	**11.** $t^2-\frac{2}{3}t+?$
4. $x^2-6x+?$	**8.** $x^2+x+?$	**12.** $t^2+\frac{3}{2}t+?$

EXAMPLES

1. Solve $x^2+4x-21=0$. (1)

Solution. Transposing, $x^2+4x=21$. (2)

Adding 4 to each member of (2),

$$x^2+4x+4=25. \quad (3)$$

Then $$(x+2)^2=5^2. \quad (4)$$

Extracting the square root of each member of (4),

$$x+2=\pm 5. \quad (5)$$

Whence $$x=-2+5=3,$$

and $$x=-2-5=-7.$$

Check. Substituting 3 for x in (1), $9 + 12 - 21 = 0$, or $0 = 0$.
Substituting -7 for x in (1), $49 - 28 - 21 = 0$, or $0 = 0$.

2. Solve $3x^2 - 10x - 8 = 0$. (1)

Solution. Transposing, $$3x^2 - 10x = 8. \quad (2)$$

Dividing (2) by the coefficient of x^2, $x^2 - \frac{10x}{3} = \frac{8}{3}$. (3)

Adding $(-\frac{5}{3})^2$ to each member of (3),

$$x^2 - \frac{10x}{3} + \left(-\frac{5}{3}\right)^2 = \frac{8}{3} + \frac{25}{9} = \frac{49}{9}. \quad (4)$$

Then $$(x - \tfrac{5}{3})^2 = (\tfrac{7}{3})^2. \quad (5)$$

Extracting the square root of each member of (5),

$$x - \tfrac{5}{3} = \pm \tfrac{7}{3}.$$

Whence $$x = \tfrac{5}{3} \pm \tfrac{7}{3} = 4 \text{ and } -\tfrac{2}{3}.$$

Check. Substituting 4 for x in (1),

$$3 \cdot 4^2 - 10 \cdot 4 - 8 = 0.$$
$$48 - 40 - 8 = 0, \text{ or } 0 = 0.$$

Substituting $-\frac{2}{3}$ for x in (1),

$$3(-\tfrac{2}{3})^2 - 10(-\tfrac{2}{3}) - 8 = 0.$$
$$\tfrac{4}{3} + \tfrac{20}{3} - 8 = 0.$$
$$\tfrac{24}{3} - 8 = 0, \text{ or } 0 = 0.$$

The method of solving a quadratic equation in x illustrated in the preceding examples may be stated in the

Rule. *Transpose so that the terms containing **x** are in the first member and those which do not contain **x** are in the second.*

Divide both members of the equation by the coefficient of x^2 unless that coefficient is $+1$.

*Then add to both members the square of one half the coefficient of **x** (in the equation just obtained), thus making the first member a perfect trinomial square.*

Rewrite the equation, expressing the first member as the square of a binomial and the second member in its simplest form.

Extract the square root of both members of the equation and write the sign $\pm$ before the square root of the second member, thus obtaining two linear equations.

Solve the equation in which the second member is taken with the sign + and then solve the equation in which the second member is taken with the sign −. The results are the roots of the quadratic.

Check. *Substitute each result separately in place of x in the original equation. If the resulting equations are not obvious identities, simplify each until it becomes one.*

If, after the equation has been simplified in accordance with the first two directions of the foregoing rule, the term in the first power of x is lacking, that is, the equation is in the form $x^2 = b$, it is often called a *pure quadratic*. Such an equation may be solved immediately by the extraction of the square root of each member.

Thus, if $5x^2 - 3 = 12$, then $5x^2 = 15$, $x^2 = 3$, and $x = \pm\sqrt{3}$.

EXERCISES

Solve by completing the square, where necessary, and check as directed by the teacher.

1. $x^2 - 4x - 32 = 0$.
2. $x^2 - 2x - 15 = 0$.
3. $x^2 = 18$.
4. $x^2 - 25 = 0$.
5. $x^2 - 7x - 18 = 0$.
6. $x(x+4) - 3(x+4) = 0$.
7. $y + 6 = y^2$.
8. $4x^2 - 4x - 3 = 0$.
9. $z^2 - 5\frac{1}{2}z + 2\frac{1}{2} = 0$.
10. $\frac{5}{4} - x^2 = 2x$.
11. $2x^2 - 9x + 4 = 0$.
12. $2t^2 - 3t = 9$.
13. $3x^2 - 5 = 70$.
14. $x^2 + 8 = 4x^2 - 40$.
15. $5x = 6x^2 - 14$.
16. $x^2 - \frac{7x}{12} - 1 = 0$.
17. $6t^2 - 13t + 6 = 0$.
18. $42 + 2x^2 = -19x$.

19. Why is not equation (5), Example 1, page 270, written with the sign $\pm$ before each member?

20. $2x^2 - 5x + 1 = 0.$ (1)

Solution. Transposing, $2x^2 - 5x = -1.$ (2)

Dividing each member of (2) by 2,

$$x^2 - \tfrac{5}{2}x = -\tfrac{1}{2}. \tag{3}$$

Adding $(-\tfrac{5}{4})^2$ to each member of (3),

$$x^2 - \tfrac{5}{2}x + (-\tfrac{5}{4})^2 = -\tfrac{1}{2} + \tfrac{25}{16} = \tfrac{17}{16}. \tag{4}$$

Then $$(x - \tfrac{5}{4})^2 = \tfrac{17}{16}. \tag{5}$$

Extracting the square root of each member of (5),

$$x - \tfrac{5}{4} = \pm\sqrt{\tfrac{17}{16}} = \pm\tfrac{1}{4}\sqrt{17}. \tag{6}$$

Whence $$x = \tfrac{5}{4} \pm \tfrac{1}{4}\sqrt{17}. \tag{7}$$

Now $$\tfrac{1}{4}\sqrt{17} = \tfrac{1}{4}(4.123+) = 1.031-. \tag{8}$$

Then $$\tfrac{5}{4} + \tfrac{1}{4}\sqrt{17} = 1.25 + (1.031-) = 2.281-. \tag{9}$$

Also $$\tfrac{5}{4} - \tfrac{1}{4}\sqrt{17} = 1.25 - (1.031-) = .219+. \tag{10}$$

Check. Since (9) and (10) are not the *exact* values of x, they will, if substituted for x in (1), make its first member *nearly* but *not quite* zero. An exact check on the radical forms of the roots can be obtained by substituting from equation (7) in equation (1).

The check may be shortened by substituting both roots at the same time, as follows:

Substituting $\tfrac{5}{4} \pm \tfrac{1}{4}\sqrt{17}$ for x in $2x^2 - 5x + 1 = 0$,

$$2(\tfrac{5}{4} \pm \tfrac{1}{4}\sqrt{17})^2 - 5(\tfrac{5}{4} \pm \tfrac{1}{4}\sqrt{17}) + 1 = 0.$$

$$2(\tfrac{25}{16} \pm \tfrac{5}{8}\sqrt{17} + \tfrac{17}{16}) - 5(\tfrac{5}{4} \pm \tfrac{1}{4}\sqrt{17}) + 1 = 0.$$

$$\tfrac{25}{8} \pm \tfrac{5}{4}\sqrt{17} + \tfrac{17}{8} - \tfrac{25}{4} \mp \tfrac{5}{4}\sqrt{17} + 1 = 0.$$

The radical terms vanish because the two upper signs before them must first be taken together, and then the two lower signs.

Therefore $$\tfrac{25}{8} + \tfrac{17}{8} - \tfrac{50}{8} + \tfrac{8}{8} = 0, \text{ or } \tfrac{50}{8} - \tfrac{50}{8} = 0.$$

In quadratic equations like the preceding the radical forms of the roots are often sufficient; at other times values to two

or three decimal places are necessary. Unless otherwise directed, obtain only the radical forms of irrational roots.

In the following ten exercises obtain correct to three decimal places the values of any radical answers which may occur:

21. $p^2+10p+17=0.$

22. $x^2+2x-4=0.$

23. $x^2-2x-3\frac{1}{2}=0.$

24. $3x^2-6x+1=0.$

25. $9x=5x^2-2.$

26. $1-4v^2=2v.$

27. $x^2-6\sqrt{3}x=-9.$

28. $q^2-3q\sqrt{2}+4=0.$

29. $x^2+5=6-4x-3x^2.$

30. $3+\dfrac{1}{3p}=3p.$

31. $x^4-4x^2+3=0.$

This is not a quadratic equation, but many equations of this form can be solved by completing the square.

Solution.

$$x^4-4x^2+3=0.$$
$$x^4-4x^2=-3.$$
$$x^4-4x^2+4=-3+4=1.$$
$$x^2-2=\pm 1.$$
$$x^2=3 \text{ and } 1.$$
$$x=\pm\sqrt{3} \text{ and } \pm 1.$$

Check as usual.

32. $x^4-5x^2+4=0.$

33. $9x^4-13x^2+4=0.$

34. $9x^4-82x^2+9=0.$

35. $x^4-10x^2+24=0.$

36. $4x^4-13x^2+3=0.$

37. $16x^4-81x^2+5=0.$

NOTE. The student has undoubtedly noticed that quadratic equations have two roots. Mathematicians attempted for many years to prove that cubic equations always have three roots and that equations of the fourth degree have four roots. This was finally done in the seventeenth century by the Italians Cardan and Ferrari.

That any equation in one unknown has a number of roots equal to its degree was first proved by K. F. Gauss. He gave three distinct proofs of this fact, although no one before his time had been able to prove it at all. Since the time of Gauss hundreds of proofs have been given by mathematicians in all parts of the world.

The mathematical researches which have been stimulated and carried to a high degree of completeness by the study of this problem

have been very numerous. This situation illustrates the important fact that when a scholar takes up a scientific study with the intention of reaching a definite result, the indirect results of his investigations are frequently even more important than the solution of the main problem which he had set himself.

Few scientists possess the vision to determine surely at the beginning of an investigation all of the directions in which it may lead. The distinction, however, between a great man and a little one largely lies in the instinct which the great man has to treat subjects which prove to be fruitful.

REVIEW EXERCISES

Solve, and check as directed by the teacher:

1. $12x^2 + 7x = 5.$
2. $(5x - 2)(x - 6) = (2x - 12)(2x - 4).$
3. $4x^2 + 15x = -9.$
4. $x + 1 = 20x^2.$
5. $25y^2 - 20y - 12 = 0.$
6. $6x + x^2 + 3 = 2 - 3x^2 + 10x.$
7. $(3x + 6)(x - 3) = (2x + 1)(x - 2).$
8. $\frac{x}{3} + 4 - \frac{15}{x} = 0.$
9. $\frac{1}{3} + \frac{a}{9} - \frac{2}{a} = 0.$
10. $\frac{3x}{2} + \frac{1}{2} = \frac{1}{3x}.$
11. $x = \frac{3}{x + 2}.$
12. $\frac{x^2}{x - 5} + \frac{5}{2} = 0.$
13. $\frac{2}{t - 2} - \frac{3t}{2} = 0.$
14. $\frac{x}{5 - x} + \frac{5x}{12} = \frac{3}{2}.$
15. $\frac{x}{x - 2} - \frac{5}{2} = \frac{2 - x}{x}.$
16. $\frac{2}{s - 2} - \frac{1}{s + 2} = -\frac{15}{8}.$
17. $\frac{3 + x}{4 + x} - \frac{1}{12} = \frac{5 - x}{6 - x}.$
18. $\frac{1 + x}{2 + x} - \frac{1}{6} = \frac{3 - x}{5 - x}.$
19. $\frac{x + 2}{x - 4} + \frac{x - 1}{x + 5} = -1.$
20. $\frac{x - 2}{x + 2} - \frac{x - 1}{x + 1} = 6.$
21. $(3x - 2)(x - 3) = (x + 1)(x + 4).$
22. $x^4 - 10x^2 + 16 = 0.$
23. $x^4 - 8x^2 + 15 = 0.$
24. $6x^4 - 19x^2 + 15 = 0.$
25. $8x^4 - 18x^2 + 9 = 0.$

PROBLEMS

(Reject all answers which do not satisfy the conditions of the problems.)

1. The square of a certain number plus twice the number itself is 8. Find the number.

2. If from twice the square of a certain number the number itself be taken, the remainder is 28. Find the number.

3. Find two consecutive odd numbers whose product is 675.

4. Find three consecutive even numbers whose sum is $\frac{1}{2}$ the product of the first two.

5. A rectangular field is 14 rods longer than it is wide. Its area is 20 acres (1 acre = 160 square rods). Find the dimensions of the field.

6. The sum of a certain number and its reciprocal is $2\frac{1}{6}$. Find the number.

7. The area of a triangular field is $4\frac{1}{2}$ acres. The base is 36 rods longer than the altitude. Find the base and the altitude.

8. Two square fields together contain 40 acres. A side of one is 16 rods longer than a side of the other. Find the side of each.

9. The hypotenuse of a right triangle is 41 feet. One leg is 31 feet shorter than the other. Find the legs.

10. One leg of a right triangle is $\frac{3}{4}$ as long as the other. The hypotenuse is 30. Find the legs.

11. The area of a square in square feet and its perimeter in inches are expressed by the same number. Find the side.

12. The dimensions of a certain rectangle and the longest straight line which can be drawn on its surface are represented in inches by three consecutive numbers. Find its dimensions.

13. The edges of two cubical bins differ by one yard. Their volumes differ by 127 cubic yards. Find the edge of each bin.

14. The rates of two trains differ by 6 miles per hour. The faster requires 1 hour less time to run 252 miles. Find the rate of each train.

15. An automobile made a round trip of 180 miles in 11 hours. On the return, the rate was increased 3 miles per hour. Find the rate each way.

16. A page of a certain book is 2 inches longer than it is wide. The printed portion covers half the area of the page and the margin is 1 inch wide. Find the length and width of the page.

17. The price of oranges being raised 10 cents per dozen, one gets 6 fewer oranges for a dollar. Find the original price.

18. Two pumps together can fill a standpipe in 30 minutes. One pump alone requires 32 minutes less time than the other. Find the time each requires alone.

19. How long will it take a stone to reach the ground if dropped from an elevation of 1600 feet? (See Exercise 45, p. 17.)

20. A man drops a stone over a cliff and hears it strike the ground below $6\frac{1}{2}$ seconds later. If sound travels 1152 feet per second, find the height of the cliff.

116. Quadratic equations with literal coefficients. Such equations are solved as in the following example.

EXAMPLE

Solve for x the equation $x^2 - 6\,ax - 7\,a^2 = 0$, and check the result.

Solution. Transposing, $x^2 - 6\,ax = 7\,a^2$.

Completing the square, $x^2 - 6\,ax + 9\,a^2 = 16\,a^2$.

Then $(x - 3\,a)^2 = 16\,a^2$.

Extracting the square root, $x - 3\,a = \pm 4\,a$.

Whence, transposing and combining, $x = 7\,a$, and $-a$.

Check. $(7\,a)^2 - 6\,a \cdot 7\,a - 7a^2 = 49\,a^2 - 42\,a^2 - 7\,a^2 = 0$, or $0 = 0$.

$(-a)^2 - 6\,a(-a) - 7\,a^2 = a^2 + 6\,a^2 - 7\,a^2 = 0$, or $0 = 0$.

EXERCISES

Solve for x and check:

1. $x^2 + 2bx = 3b^2$.
2. $x^2 + 4cx = 5c^2$.
3. $x^2 - 8cx - 9c^2 = 0$.
4. $x^2 + ax - 2a^2 = 0$.
5. $x^2 - 2ax = 1 - a^2$.
6. $x^2 - a^2 = 0$.
7. $a^2x^2 = b^2$.
8. $x^2 + bx = 2b^2$.
9. $x^2 - 12a = 3ax - 4x$.
10. $2x^2 - 3hx - 2h^2 = 0$.
11. $10x^2 = 3a^2 - ax$.
12. $2x^2 - 5hx + 2h^2 = 0$.
13. $ax^2 - 5x - 2ax + 10 = 0$.
14. $cx^2 + 2x = 3cx + 6$.
15. $ax^2 + a = a^2x + x$.
16. $cx^2 + ax = 2cx + 2a$.

History of the quadratic equation. Though the development of the method of solving quadratic equations is closely connected with the general growth of algebra, yet it is possible to indicate rather briefly the most important steps in the process.

The first writer on formal algebra was Diophantos, who lived at Alexandria, in Egypt, about A.D. 275. Most of his work that is preserved is devoted to the solution of problems that lead to equations. So far as we know he was the first to indicate the unknown number by a single letter, in this respect being far in advance of many mathematicians who lived much later. It is a little remarkable, in fact, that so able and original a man as Diophantos should have exerted so little influence on his successors. He solved his quadratic equations by a method not unlike that of completing the square, but his imperfect knowledge of the nature of numbers made it impossible for him to understand the entire significance of the process. Though he made every effort not to consider equations whose roots were not positive integers, sometimes they would creep in, and under such circumstances, when his method led him to a negative or irrational root, he rejected the whole equation as absurd or impossible. Even when both of the roots were positive he took only the one afforded by the positive sign in the formula for solving a quadratic.

The difficulties of Diophantos are typical of those encountered by mathematicians for the next fifteen hundred years. The difficulty lay not in finding a formal method of solving the equation, but in understanding the result after it was obtained. The meanings of

negative and of imaginary numbers have been two of the most difficult of all mathematical ideas for men to grasp.

Five or six hundred years later the Hindus devised a general solution of the quadratic, but their chief advance over Diophantos lay in the fact that they did not regard an equation whose roots were negative as necessarily absurd, but merely rejected the negative result with the remark, "It is inadequate; people do not approve of negative roots." The Hindus, however, did realize that a quadratic equation sometimes has two roots, a fact that Diophantos never comprehended.

No material gain in the understanding of the solutions of the quadratic can be found until the seventeenth century. The keenest mathematicians of the sixteenth century, like Cardan and Vieta, rejected negative roots, though by this time irrational roots were admitted. In fact, in 1544 Stifel, a German, published an algebra in which irrational numbers are included among the numbers proper. But he affirms that except in the case where a quadratic equation has two positive roots no equation has more than one root. It was not until the work of Descartes and Gauss became widely known that the nature of the roots of all kinds of quadratic equations was completely understood.

117. Systems involving a linear and a quadratic equation. A **quadratic equation** in two unknowns contains one or more terms of the second degree, but no term of higher degree in those unknowns. Every system of equations in two unknowns in which one equation is **linear** and the other **quadratic** *can be solved by the method of* **substitution.**

EXAMPLE

Solve the system
$$\begin{cases} x^2 + y^2 = 5, & (1) \\ x - y = 1. & (2) \end{cases}$$

Solution. Solving (2) for x, $x = 1 + y$. (3)

Substituting $1 + y$ for x in (1), $(1 + y)^2 + y^2 = 5$. (4)

From (4), $y^2 + y - 2 = 0$. (5)

Solving (5), $y = 1$ or -2.

Substituting 1 for y in (3), $x = 1 + 1 = 2.$

Substituting -2 for y in (3), $x = 1 - 2 = -1.$

Therefore $\begin{cases} x = 2 \\ y = 1 \end{cases}$ and $\begin{cases} x = -1 \\ y = -2 \end{cases}$ are the two sets of roots.

Check. Substituting 2 for x and 1 for y in $\begin{cases} (1), & 4 + 1 = 5, \\ (2), & 2 - 1 = 1. \end{cases}$

Substituting -1 for x and -2 for y in $\begin{cases} (1), & 1 + 4 = 5, \\ (2), & -1 + 2 = 1. \end{cases}$

The similarity between this method and that for the solution of linear systems by substitution should be carefully noted.

The method of the preceding solution is stated in the

Rule. *Solve the linear equation for one unknown in terms of the other.*

Substitute this value in the quadratic equation and solve the resulting equation.

Substitute each of the roots of the quadratic equation thus found in the linear equation, and solve, thus obtaining two sets of roots of the simultaneous system.

Check, as usual, in both equations.

EXERCISES

Solve the following systems, pair results, and check each set of roots:

1. $x^2 + y = 8,$
   $2x + y = 5.$
2. $x + y = 5,$
   $x^2 + y^2 = 13.$
3. $x^2 + y^2 = 25,$
   $3x - 4y = 0.$
4. $x^2 + y^2 = 25,$
   $x - 3y = 5.$
5. $xy = 12,$
   $2x + y - 10 = 0.$
6. $x^2 + 3xy = 25,$
   $2x + y = 10.$
7. $4p + 5q = 6,$
   $pq + 2 = 0.$
8. $4x - 3y = 30,$
   $xy + 12 = 0.$
9. $3A + 2B = 5,$
   $AB + 3 = 6A.$
10. $x^2 + xy = 5,$
    $x - y = 9.$
11. $x^2 + y^2 + y = 46,$
    $2x + y = 10.$
12. $x^2 + 3xy + y^2 = 22,$
    $2x = y.$
13. $x^2 + y^2 - 4x - 2y = 20,$
    $3x - y = 10.$

PROBLEMS

(Reject all results which do not satisfy the conditions of the problem.)

1. Find two numbers whose difference is 3 and the difference of whose squares is 45.

2. The sum of two numbers is 18 and the sum of their squares is 170. Find the numbers.

3. A rectangular field is 39 rods longer than it is wide and its area is 10 acres. Find the length and the width.

4. The difference of the areas of two squares is 208 square feet and the difference of their perimeters is 32 feet. Find a side of each square.

5. The perimeter of a rectangle is 92 feet and its area is 504 square feet. Find the length and the width.

6. The base of a triangle is 5 inches longer than its altitude. Its area is $1\frac{1}{24}$ square feet. Find the base and the altitude of the triangle.

7. The perimeter of a rectangle is $3c$ and its area is $\frac{c^2}{2}$. Find its dimensions.

8. Do positive integers differing by 3 exist such that the sum of their squares is 115? If so, find them.

9. If a two-digit number be multiplied by the sum of its digits, the product is 576. If three times the sum of its digits be added to the number, the result is expressed by the digits in reverse order. Find the number.

10. The annual income from a certain investment is \$48. If the principal were \$200 more and the rate of interest 1% less, the annual income would be \$2 more. Find the principal and the rate.

11. A wheelman leaves A and travels north. At the same time a second wheelman, who travels 50% faster than the first, leaves a point 3 miles east of A and travels east. An hour after starting, the distance between them is 17 miles. Find the rate of each.

CHAPTER XXII

RATIO AND PROPORTION

118. Ratio. The **ratio** of one number, a, to a second number, b, is the quotient obtained by dividing the first by the second, that is, $\frac{a}{b}$. This ratio is also written $a:b$.

It follows from the above that all ratios of numbers are fractions and that all fractions may be regarded as ratios.

Thus $\frac{3}{2}$, $\frac{c}{2x}$, $\frac{a+b}{a-b}$, and $\frac{\sqrt{2}}{\sqrt[3]{5}}$ are ratios as well as fractions.

We may speak of the ratio of two concrete numbers if they have a common unit of measure. The ratio of 5 feet to 3 feet is $\frac{5}{3}$, the common unit of measure being 1 foot. Obviously no ratio exists between 5 years and 3 feet.

If we say a piece of paper contains 54 square inches, we are expressing by the number 54 the ratio of the surface of the paper to the surface of a square whose side is one inch.

Every measurement, then, is the determination of a ratio, either exact or approximate.

EXERCISES

Simplify the following ratios by writing them as fractions and reducing the fractions to their lowest terms:

1. $6:9$. **3.** $5:10$. **5.** $20x:12x^2$. **7.** 3 hours : 150 minutes.

2. $12:8$. **4.** $14:7$. **6.** $17\frac{1}{4}:3\frac{5}{6}$. **8.** 900 pounds : $1\frac{1}{2}$ tons.

9. Separate 45 into two parts which are in the ratio of $7:8$.

HINT. Let $7x$ = one part; then $8x$ = the other.

10. Separate 121 into two parts which are in the ratio $3:8$.

11. Separate 152 into two parts which are in the ratio $2\frac{1}{2}:7$.

12. What number added to both terms of the ratio $5:9$ gives a result equal to the ratio $36:54$?

13. What number subtracted from both terms of the ratio $11:12$ gives a result equal to the ratio $35:40$?

14. If x is a positive number, which is the greater ratio: $\frac{3}{4}$ or $\frac{3+x}{4+x}$? $\frac{x}{x+1}$ or $\frac{2x}{2x+1}$?

HINT. Reduce the two fractions in each part to respectively equivalent fractions having a common denominator, and then compare numerators.

15. From your answers to Exercise 14 state the change that is produced in the value of a proper fraction (sometimes called a ratio of less inequality) by adding a positive number to both of its terms.

16. If x is a positive number, which is the greater ratio: $\frac{7}{5}$ or $\frac{7+x}{5+x}$? $\frac{1+x}{x}$ or $\frac{1+2x}{2x}$?

17. From your answers to Exercise 16 state the change that is produced in the value of an improper fraction (sometimes called a ratio of greater inequality) by adding a positive number to both of its terms.

119. Proportion. A **proportion** is a statement of equality between two ratios.

Thus $\frac{1}{2}=\frac{3}{6}$, $\frac{4}{5}=\frac{8}{10}$, and $\frac{12}{20}=\frac{3}{5}$ are proportions, for the equality of the ratios is evident in each case.

The four numbers 1, 2, 3, and 6 are said to be in proportion, for the ratio of the first pair equals the ratio of the second pair. In general, the numbers a, b, c, and d are in proportion if $a:b=c:d$. (1)

In (1), a and d (the first and fourth terms) are called **extremes**, and b and c are called **means**.

Since a proportion is an equality between two ratios (fractions), it is therefore an equation. Hence *any operation which may be performed on an equation may be performed on a proportion.* (See Axioms, pp. 39–40.)

Thus, in the proportion $\frac{a}{b}=\frac{c}{d}$ both members may be multiplied by bd, giving $ad=bc$. Here the first member is the product of the extremes of the proportion, and the second member is the product of the means.

Therefore, *in any proportion the product of the extremes equals the product of the means.*

EXERCISES

1. Form proportions from the following by supplying the missing terms: (*a*) $\frac{4}{5}=\frac{8}{?}$; (*b*) $\frac{15}{20}=\frac{3}{?}$; (*c*) $\frac{7}{21}=\frac{?}{3}$; (*d*) $\frac{45}{60}=\frac{?}{4}$; (*e*) $\frac{12}{?}=\frac{2}{3}$; (*f*) $\frac{50}{?}=\frac{5}{1}$; (*g*) $\frac{?}{18}=\frac{2}{9}$; (*h*) $\frac{12}{32}=\frac{?}{8}$.

Solve the following for x:

2. $2\frac{1}{4}:9=x^2:12$.

3. $\frac{x-1}{2x-3}=\frac{5}{9}$.

4. $(x-3):5=7:\frac{x}{2}$.

5. $1:4=1:\frac{1}{8x}$.

6. $\frac{13}{3x}=\frac{x+11}{4x-2}$.

7. $\frac{4}{x}=\frac{x}{25}$.

8. $\frac{16}{1}=\frac{144}{x^2}$.

9. $x:s=t:f$.

10. $f:x=t:f$.

11. $f:s=t:x$.

12. $f:x=x:1$.

120. Fourth, mean, and third proportionals. A **fourth proportional** to three numbers is the fourth term of a proportion in which the three numbers appear in the order in which they are given as the first, second, and third terms of the proportion respectively.

Thus 6 is a fourth proportional to 2, 3, and 4, for $\frac{2}{3}=\frac{4}{6}$. Again, 20 is a fourth proportional to 3, 5, and 12, for $\frac{3}{5}=\frac{12}{20}$.

In general, *f is the fourth proportional to three numbers a, b, and c if*

$$\frac{a}{b}=\frac{c}{f}.$$

A **mean proportional** between two numbers is the second or the third term of a proportion in which the means are identical.

Thus 2 is a mean proportional between 1 and 4, for $1:2=2:4$. Also -2 is a mean proportional between 1 and 4, for $1:-2=-2:4$.

In general, *m is the mean proportional between two numbers a and b if*

$$\frac{a}{m}=\frac{m}{b}, \quad \text{or if} \quad m=\pm\sqrt{ab}.$$

A **third proportional** to two numbers is the last term of a proportion which has for its first term the first of the given numbers and for both second and third terms the second given number.

Thus 16 is the third proportional to 9 and 12, for $\frac{9}{12}=\frac{12}{16}$. Again, 25 is the third proportional to 4 and 10, for $\frac{4}{10}=\frac{10}{25}$.

In general, *t is the third proportional to the numbers a and b if*

$$\frac{a}{b}=\frac{b}{t}.$$

EXERCISES

Find a fourth proportional to:

1. 2, 5, and 8.

HINT. $\frac{2}{5}=\frac{8}{f}$.

2. 1, 5, and 4.

3. 7, 10, and -21.

4. 6, 9, and 12.

5. 6, 8, and t.

6. 2, $6x$, and $12x$.

7. $-x$, x^2, and $2x^3$.

Find the mean proportionals between:

8. 1 and 9.

HINT. $\frac{1}{m}=\frac{m}{9}$.

9. 4 and 25.

10. 9 and $\frac{1}{4}$.

11. $\frac{1}{24}$ and $\frac{1}{6}$.

12. $\frac{7}{2}$ and 2.

13. $\sqrt{2}$ and $\sqrt{50}$.

14. $\sqrt{\frac{1}{2}}$ and $\sqrt{8}$.

Find the third proportionals to:

15. 1 and 2.

HINT. $\frac{1}{2} = \frac{2}{t}$.

16. 5 and 15.

17. 3 and -7.

18. 2 and $\sqrt{5}$.

19. 3 and $\sqrt{6}$.

20. 3 and $2\sqrt{6}$.

21. 7 and $2\sqrt{14}$.

22. x and $-2x$.

23. a and ab.

24. c and $\sqrt{ad}$.

121. Proportions from equal products. That the numbers which occur in a pair of equal products may be used in various ways as the terms of a proportion is illustrated in the following:

From $2 \cdot 6 = 4 \cdot 3$ we may write the proportions $\frac{6}{3} = \frac{4}{2}$, $\frac{2}{4} = \frac{3}{6}$, $\frac{6}{4} = \frac{3}{2}$, and $\frac{2}{3} = \frac{4}{6}$. Each of these four statements is a proportion, for in each case the equality of the ratios is evident.

In general, *if the product of two numbers* **a** *and* **d** *equals the product of two other numbers* **b** *and* **c**, *one pair may be made the extremes and the other pair the means of a proportion;*

that is, if
$$ad = bc,$$
then
$$\frac{a}{b} = \frac{c}{d}, \text{ or } \frac{a}{c} = \frac{b}{d}.$$

Proof. If $a \cdot d = b \cdot c$ is divided by bd, we obtain

$$\frac{a\cancel{d}}{b\cancel{d}} = \frac{\cancel{b}c}{\cancel{b}d}, \quad \text{or} \quad \frac{a}{b} = \frac{c}{d}. \tag{1}$$

If $a \cdot d = b \cdot c$ is divided by cd, we obtain

$$\frac{a}{c} = \frac{b}{d}. \tag{2}$$

The transformation from (1) to (2) above (where the means are interchanged) is called **alternation.**

If $a \cdot d = b \cdot c$ is divided by ac, we obtain

$$\frac{b}{a} = \frac{d}{c}. \tag{3}$$

The transformation from (1) to (3) (where the fractions are inverted) is called **inversion.**

EXERCISES

Write as a proportion in three ways:

1. $2 \cdot 3 = 6 \cdot 1$.
2. $3 \cdot 8 = 4 \cdot x$.
3. $a \cdot b = c \cdot 1$.

Write as a proportion having f for the fourth term:

4. $5 \cdot f = 3 \cdot 2$.
5. $5 \cdot 6 = 2 \cdot f$.
6. $f = \frac{ad}{b}$.

Write the following proportions by alternation:

7. $\frac{3}{4} = \frac{9}{12}$.
8. $\frac{2}{a} = \frac{3}{b}$.
9. $\frac{5}{x} = \frac{m}{n}$.
10. $\frac{m}{n} = \frac{r}{s}$.

Write the following proportions by inversion:

11. $\frac{3}{4} = \frac{9}{12}$.
12. $\frac{2}{x} = \frac{8}{3}$.
13. $\frac{n}{m} = \frac{s}{r}$.
14. $\frac{P_1}{P_2} = \frac{W_2}{W_1}$.

122. Proportion by addition. The dividend, or numerator, in a ratio is called the **antecedent**, and the divisor, or denominator, is called the **consequent**.

If in the proportion $\frac{2}{3} = \frac{4}{6}$ we add each antecedent to its consequent and divide the sum thus obtained in each case by the consequent of its ratio, we have

$$\frac{2+3}{3} = \frac{4+6}{6}, \quad \text{or} \quad \frac{5}{3} = \frac{10}{6},$$

which is a proportion. In the same manner we obtain from the proportion $\frac{3}{7} = \frac{15}{35}$ the proportion $\frac{10}{7} = \frac{50}{35}$.

In general, *if four numbers* **a**, **b**, **c**, *and* **d** *are in proportion, they are in proportion by* ***addition***; *that is, the sum of the first two terms is to the second term as the sum of the last two terms is to the fourth term.*

Proof. Let
$$\frac{a}{b} = \frac{c}{d}. \qquad (1)$$

Adding 1 to each member, we have

$$\frac{a}{b} + 1 = \frac{c}{d} + 1, \quad \text{or} \quad \frac{a+b}{b} = \frac{c+d}{d}. \qquad (2)$$

Here (2) is said to be obtained from (1) by *addition*.

EXERCISES

Write by addition and test results in the numerical exercises:

1. $\frac{2}{5} = \frac{4}{10}$.
2. $\frac{1}{3} = \frac{9}{27}$.
3. $\frac{12}{15} = \frac{32}{40}$.
4. $\dfrac{4}{11} = \dfrac{22}{x}$.
5. $x : y = 2 : 3$.
6. $5 : 7 = c : d$.
7. $f : s = t : 1$.
8. $A_1 : A_2 = r_1^2 : r_2^2$.
9. If $\dfrac{a}{b} = \dfrac{c}{d}$, show that $\dfrac{a+b}{a} = \dfrac{c+d}{c}$.

123. Proportion by subtraction. If in the proportion $\frac{7}{5} = \frac{21}{15}$ we subtract each consequent from its antecedent and divide the remainder thus obtained in each case by the consequent of its ratio, we have

$\dfrac{7-5}{5} = \dfrac{21-15}{15}$, or $\dfrac{2}{5} = \dfrac{6}{15}$, which is obviously a proportion.

In general, *if four numbers* **a**, **b**, **c**, *and* **d** *are in proportion, they are in proportion by* **subtraction**; *that is, the first antecedent minus its consequent is to that consequent as the second antecedent minus its consequent is to that consequent.*

Proof. Let
$$\frac{a}{b} = \frac{c}{d}. \tag{1}$$

Subtracting 1 from each member, we have
$$\frac{a}{b} - 1 = \frac{c}{d} - 1, \quad \text{or} \quad \frac{a-b}{b} = \frac{c-d}{d}. \tag{2}$$

Here (2) is said to be obtained from (1) by *subtraction*.

EXERCISES

Write by subtraction and test resulting proportions in the numerical exercises:

1. $\frac{6}{3} = \frac{2}{1}$.
2. $\frac{5}{4} = \frac{10}{8}$.
3. $\frac{9}{12} = \frac{15}{20}$.
4. $\frac{18}{13} = \frac{54}{39}$.
5. $2 : 5 = 3 : x$.
6. $7 : 8 = m : n$.
7. $a : b = m : n$.
8. $f : s = t : 1$.
9. If $a : b = c : d$, show that $(a - b) : a = (c - d) : c$.

124. Proportion by addition and subtraction. If we write the proportion $\frac{8}{5}=\frac{32}{20}$ by addition, we have

$$\frac{13}{5}=\frac{52}{20}. \tag{1}$$

If we write the same proportion by subtraction, we have

$$\frac{3}{5}=\frac{12}{20}. \tag{2}$$

Now (1) ÷ (2) gives the proportion

$$\frac{13}{3}=\frac{52}{12}.$$

In general, *if four numbers* **a**, **b**, **c**, *and* **d** *are in proportion, they are in proportion by* ***addition and subtraction***; *that is, the sum of the first antecedent and its consequent is to their difference as the sum of the second antecedent and its consequent is to their difference.*

Proof. Let
$$\frac{a}{b}=\frac{c}{d}. \tag{1}$$

Then
$$\frac{a+b}{b}=\frac{c+d}{d} \text{ (addition)}, \tag{2}$$

and
$$\frac{a-b}{b}=\frac{c-d}{d} \text{ (subtraction)}. \tag{3}$$

Dividing (2) by (3),
$$\frac{a+b}{a-b}=\frac{c+d}{c-d}. \tag{4}$$

Proportion (4) is said to be obtained from proportion (1) by *addition and subtraction.*

Proportions by addition, subtraction, and addition and subtraction are often called proportions *by composition*, *division*, and *composition and division* respectively.

EXERCISES

1. Write Exercises 1–3, p. 288, by addition and subtraction.

Using addition and subtraction, solve the following for x:

2. $\dfrac{5x+6}{5x-6}=\dfrac{3}{2}.$

3. $\dfrac{2x+3}{2x-3}=\dfrac{8+3}{8-3}.$

4. $\dfrac{x+\sqrt{2}}{x-\sqrt{2}}=\dfrac{3-\sqrt{2}}{3+\sqrt{2}}.$

5. $\dfrac{\sqrt{x}+2}{\sqrt{x}-2}=\dfrac{\sqrt{2x}+3}{\sqrt{2x}-3}.$

6. If $a:b=c:d$, show that $3a:b=3c:d$.

7. If $a:b=c:d$, show that $(3a+b):b=(3c+d):d$.

8. If $a:b=c:d$, show that $(a+2b):b=(c+2d):d$.

125. A series of equal ratios. We now proceed to prove that *in a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent;*

that is, if $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, then $\dfrac{a+c+e}{b+d+f}=\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$.

Proof. Setting each of the given equal ratios above equal to r,

$$\frac{a}{b}=r;\ \frac{c}{d}=r;\ \text{and}\ \frac{e}{f}=r. \tag{1}$$

Then from (1),
$$a=b\cdot r, \tag{2}$$
$$c=d\cdot r, \tag{3}$$
$$e=f\cdot r. \tag{4}$$

Adding (2), (3), and (4),
$$a+c+e=br+dr+fr. \tag{5}$$

Factoring in (5),
$$a+c+e=(b+d+f)r. \tag{6}$$

Therefore
$$\frac{a+c+e}{b+d+f}=r. \tag{7}$$

Hence by (1), and (7),
$$\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}.$$

EXERCISES

Test the truth of the preceding statement in the following:

1. $\frac{2}{3}=\frac{4}{6}=\frac{10}{15}$. **2.** $\dfrac{a}{2}=\dfrac{3a}{6}=\dfrac{4ab}{8b}$. **3.** $\frac{5}{6}=\frac{10}{12}=\frac{15}{18}$.

4. $2:(c+d)=2a:(ac+ad)=2c:(c^2+cd)$.

5. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, show that $\dfrac{3a+4c+5e}{3b+4d+5f}=\dfrac{a}{b}$.

PROBLEMS

1. A tract of land is purchased by two men jointly, one contributing \$2500 and the other \$3500. During a given year oil wells on the tract produced in royalties \$2400. How much of this sum should each receive?

2. Three men bought a piece of property, contributing \$1000, \$1500, and \$2000 respectively. If in selling the property the owners gain \$900, how much should each receive?

3. The perimeter of a triangle is 39. Two sides are 10 and 16. The other side is divided into two parts which are in the ratio of these two. Find the parts of the third side.

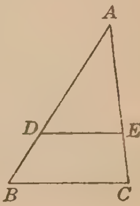
4. The sides of a triangle are 15, 20, and 25. The side 20 is divided into parts which are proportional to the other two sides. Find these two parts.

Fact from Geometry. If one triangle is similar to another, the sides of the first taken in any order are proportional to the sides of the second taken in the same order.

5. The sides of a triangle are 15, 21, and 33 respectively. In a similar triangle the side which corresponds to 21 in the first triangle is 14. Find the other two sides. Compare the ratio of any pair of corresponding sides with that of the perimeters.

6. The sides of a triangle are 12, 28, and 36. The perimeter of a similar triangle is $\frac{3}{4}$ that of the given triangle. Find the sides of the second triangle.

Facts from Geometry. A line parallel to one side of a triangle divides the other two sides into four parts which are proportional, and the triangle cut off is similar to the first.



Thus, in triangle ABC, if DE is parallel to side BC, then

$$\frac{BD}{DA} = \frac{CE}{EA}, \quad \text{and} \quad \frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}.$$

7. If in the triangle ABC on page 291,

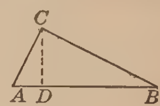
(a) $AD = 12$, $DB = 9$, and $AE = 8$, find EC.

(b) $AD = 12$, $AB = 21$, and $DE = 8$, find BC.

(c) $DB = 9$, $AD = 12$, and $AC = 14$, find AE.

8. A flagstaff casts a shadow 100 feet long at the same time that a man 5 feet 8 inches high casts a shadow 14 feet and 2 inches long. How high is the flagstaff?

9. In the triangle ABC, having a right angle at C, CD is perpendicular to AB and so divides AB that CD is a mean proportional between AD and DB.



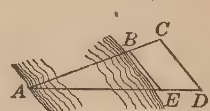
(a) If $AD = 3$, and $BD = 12$, find CD.

(b) If $AD = 4$, and $CD = 6$, find DB.

(c) If $AB = 26$, and $CD = 12$, find AD and DB.

(d) If $AB = 34$, and $CD = 15$, find AD and DB.

10. The distance AB between two points on opposite banks of a river is wanted. Stakes were set at E, B, D, and C so that BE was parallel to CD and so that ABC and AED were straight lines. The measured values of DC, CB, and BE were 560 feet, 250 feet, and 360 feet respectively. What was the computed value of BA?



11. Two men start at the same time and travel in opposite directions at rates in the ratio of 14:16. If in 14 hours they are 105 miles apart, find the rate of each.

12. The diameters of the earth and of the sun are in the ratio 1:108. Given that the volumes of spheres are proportional to the cubes of their respective diameters, find the ratio of the volumes of the earth and the sun.

13. Given that the surfaces of spheres are proportional to the squares of their respective radii, find the ratio of the surfaces of the earth and the sun.

SUPPLEMENTARY TOPICS

126. Highest common factor. In arithmetic the greatest common divisor of two or more integers is the greatest integer which is exactly contained in each of the given integers.

This greatest common divisor for two or more integers, or highest common factor (H.C.F.), as the corresponding idea is commonly designated in algebra, can often be obtained by inspection; that is, without writing the integers as the indicated product of their prime factors.

Thus the H.C.F. of 5 and 15 is 5; of 30 and 42 is 6; of $6a$ and $8a^2$ is $2a$.

The degree of a polynomial is the same as that of its term of highest degree. (See page 75.)

Thus $4a^2xy + 5ax^5 - 3a^3xyz^2$ is of the seventh degree.

The **highest common factor** (H.C.F.) of two or more monomials or polynomials is the expression of highest degree, with the greatest numerical coefficient, which is an exact divisor of each.

Thus the H.C.F. of $28a^2b^3$ and $42a^2b^2$ is $14a^2b^2$. The H.C.F. of $x^3 - 4x$ and $x^3 - 5x^2 + 6x$ is $x(x-2)$, or $x^2 - 2x$.

EXAMPLES

1. Find the H.C.F. of $72x^3y^5z$, $90x^2y^3z^4$, and $108x^4y^4z^3$.

Solution. Factoring, we have

$$72x^3y^5z = 2^3 \cdot 3^2 \cdot x^3y^5z,$$
$$90x^2y^3z^4 = 2 \cdot 3^2 \cdot 5x^2y^3z^4,$$
$$108x^4y^4z^3 = 2^2 \cdot 3^3 \cdot x^4y^4z^3.$$

Here the highest power of 2 common to each expression is the first; of 3, the second; of x, the second; of y, the third; and of z, the first. Therefore the H.C.F. of the three expressions is $2 \cdot 3^2 \cdot x^2y^3z$, which equals $18x^2y^3z$.

2. Find the H.C.F. of $9x^4 - 36x^2$ and $3x^7 - 12x^6 + 12x^5$.

Solution. Factoring, we have

$$9x^4 - 36x^2 = 3^2x^2(x+2)(x-2),$$
$$3x^7 - 12x^6 + 12x^5 = 3x^5(x-2)^2.$$

Therefore the H.C.F. is $3x^2(x-2)$, which equals $3x^3 - 6x^2$.

The method used in the preceding solutions for finding the H.C.F. of two or more monomials or polynomials is stated in the

Rule. *Separate each expression into its prime factors. Then find the product of such factors as occur in each expression, using each prime factor the least number of times it occurs in any one expression.*

If two or more polynomials have no common factor other than 1, then 1 is the H.C.F. of the polynomials, and they are said to be prime to each other.

EXERCISES

Find the H.C.F. of the following:

1. 50, 75.
2. 24, 42.
3. 72, 60.
4. 90, 105.
5. 126, 162, 198.
6. 105, 189, 231.
7. 208, 364, 468.
8. 84, 210, 462, 588.
9. $10a^3, 4a^5, 12a^4$.
10. $24x^2, 18x^7y$.
11. $16x^4, 40x^2, 24x^3$.
12. $60a^2x, 84ax^3, 24a^6x^2y^2$.
13. $30a^2c^3, 75ac^5, 150a^4c^6d, 90a^3c^4x^2$.
14. $36x^3y^4, 90x^6y^2z, 126x^5yz, 180x^5y^2$.
15. $72xy^3z^7, 216x^2y^2z^5, 504x^2y^5z^7, 144x^2y^3z^8$.
16. $(a^2 - 4), (a^2 + 4a + 4)$.
17. $(x^2 - 9), (x^2 - x - 6)$.
18. $(c^4 + c^2 - 20), (c^3 + 5c)$.
19. $(b^2 - 1), (b^3 + 1), (2b^2 + 5b + 3)$.
20. $(8 - 4a), (8 - a^3), (4 - a^2)$.
21. $(3x^2 - 3y^2), (3x^2 - 3xy), (6x^3 - 6y^3)$.
22. $(5c^2x^2 - 20x^2), (5c^4 + 5c^3 - 30c^2), (30c^5 - 40c^4 - 40c^3)$.

127. The square of any trinomial. The multiplication

$$\begin{array}{l} a + b + c \\ \underline{a + b + c} \\ a^2 + ab + ac \\ \quad + ab \qquad\quad + b^2 + bc \\ \qquad\quad + ac \qquad\quad + bc + c^2 \\ \hline a^2 + 2\,ab + 2\,ac + b^2 + 2\,bc + c^2 \end{array}$$

gives the formula

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc.$$

This may be expressed in words as follows:

The square of any trinomial is equal to the sum of the squares of the terms, plus twice the algebraic product of each term by each term that follows it in the trinomial.

EXERCISES

Expand by the method indicated above:

1. $(a + x + 1)^2$.
2. $(m + n + 2)^2$.
3. $(r - 5 + s)^2$.
4. $(c + d - 3)^2$.
5. $(a + c + x)^2$.
6. $(c + x - 2)^2$.
7. $(a + b + 3)^2$.
8. $(x + 6 + 2\,a)^2$.
9. $(c + 2\,a + 2)^2$.
10. $(a - x + 2\,c)^2$.
11. $(a - 2\,x - c)^2$.
12. $(3\,a + x + 2\,c)^2$.
13. $(4\,a - 2\,x + c^2)^2$.
14. $(3\,a^2 + 3\,x - 1)^2$.
15. $(5\,a - 2\,c^2 - 3)^2$.

128. Factors of $x^5 \pm y^5$. By division we obtain

$$\frac{x^5 + y^5}{x + y} = x^4 - x^3y + x^2y^2 - xy^3 + y^4.$$

Hence the sum of the fifth powers of two numbers is exactly divisible by the sum of the numbers.

EXERCISES

Find the prime factors of the following:

1. $a^5 + b^5$.
2. $x^5 + a^5$.
3. $a^5 + 2^5$.
4. $x^5 + 32$.
5. $32 + c^5$.
6. $243 + x^5$.
7. $(c^2)^5 + d^5$.
8. $m^{10} + n^5$.
9. $r^5 + s^{15}$.

10. $s^{15} + t^{10}$.

11. $x^5 - y^5$.

HINT. Divide by $x - y$.

12. $a^5 - b^5$.

13. $c^5 - x^5$.

14. $m^5 - n^5$.

15. $m^5 - 2^5$.

16. $n^5 - 32$.

17. $243 - a^5$.

18. $(x^2)^5 - y^5$.

19. $a^{10} - b^5$.

20. $x^{10} - y^{10}$.

129. Operations with surds of different orders. If the indicated root of a rational number cannot be taken exactly, the radical is sometimes called a **surd**.

Surds of different orders should be reduced to respectively equivalent surds of the same order before the operations of multiplication or division are performed with them. This process is also necessary before the relative magnitudes of surds can be compared.

EXAMPLES

1. Multiply $\sqrt[3]{4}$ by $\sqrt{5}$.

Solution. $\sqrt[3]{4} \cdot \sqrt{5} = 4^{\frac{1}{3}} \cdot 5^{\frac{1}{2}} = 4^{\frac{2}{6}} \cdot 5^{\frac{3}{6}} = \sqrt[6]{4^2} \cdot \sqrt[6]{5^3} = \sqrt[6]{4^2 \cdot 5^3} = \sqrt[6]{2000}.$

2. Divide $\sqrt{2}$ by $\sqrt[3]{6}$.

Solution. $\sqrt{2} \div \sqrt[3]{6} = 2^{\frac{1}{2}} \div 6^{\frac{1}{3}} = 2^{\frac{3}{6}} \div 6^{\frac{2}{6}} = \sqrt[6]{2^3} \div \sqrt[6]{6^2}$

$$= \sqrt[6]{\frac{2^3}{6^2}} = \sqrt[6]{\frac{2}{3^2}} = \sqrt[6]{\frac{2 \cdot 3^4}{3^6}} = \frac{\sqrt[6]{162}}{3}.$$

EXERCISES

Perform the indicated operations and simplify results:

1. $\sqrt[3]{4} \cdot \sqrt{3}$.

2. $\sqrt[3]{3} \cdot \sqrt{3}$.

3. $\sqrt{3} \cdot \sqrt[3]{2}$.

4. $\sqrt[3]{9} \cdot \sqrt{3}$.

5. $\sqrt[5]{25} \cdot \sqrt{5}$.

6. $\sqrt{a} \cdot \sqrt[3]{a^2}$.

7. $\sqrt[3]{5x^2} \cdot \sqrt{10x}$.

8. $\sqrt{\frac{a}{b}} \cdot \sqrt[3]{\frac{a}{b}}$.

9. $\sqrt[3]{\frac{2a^2}{x}} \cdot \sqrt{\frac{x}{4a}}$.

10. $\sqrt{\frac{2x}{5y}} \cdot \sqrt[3]{\frac{25y}{4x}}$.

11. $\sqrt{2} \div \sqrt[3]{4}$.

12. $\sqrt[3]{6} \div \sqrt{2}$.

13. $\sqrt{10} \div \sqrt[3]{5}$.

14. $\sqrt[5]{a} \div \sqrt{a}$.

15. $\sqrt{c} \div \sqrt[5]{c^2}$.

16. $\sqrt{3a} \div \sqrt[3]{3a^2}$.

17. $\sqrt[3]{10x} \div \sqrt{2x}$.

18. $\sqrt[3]{\frac{2}{3}} \div \sqrt{\frac{2}{3}}$.

19. $\sqrt[3]{\frac{2x^2}{a^2}} \div \sqrt{\frac{x}{3}}$.

20. $\sqrt[5]{\frac{3}{2}} \div \sqrt{\frac{3}{2}}$.

Arrange the following in order of magnitude:

21. $\sqrt[3]{5}$, $\sqrt{7}$.

HINT. Reduce to respectively equivalent surds of the sixth order and compare radicands.

22. $\sqrt{5}$, $\sqrt[3]{11}$.

23. $\sqrt[3]{5}$, $\sqrt{3}$.

24. $\sqrt[3]{9}$, $\sqrt{5}$.

25. $\sqrt{12}$, $\sqrt[3]{40}$.

26. $\sqrt[5]{16}$, $\sqrt[3]{4}$.

27. $\sqrt[4]{5}$, $\sqrt[2]{2}$, $\sqrt[3]{3}$.

130. Solution of the quadratic equation by formula. If the general quadratic equation $ax^2 + bx + c = 0$ is solved by the method of completing the square, the result is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{F}$$

This expression is a general result and may be used as a formula for the solution of any quadratic equation.

If the numbers a, b, and c are such that $b^2 - 4ac$ is negative, then the formula would contain the square root of a negative number, which is a kind of number not considered in this text. In the exercises that follow, it will be assumed that only such numerical values of the letters involved are considered as will not make $b^2 - 4ac$ negative.

EXERCISES

Solve by formula, and check.

1. $4x^2 + 8x = 3$.

Solution. Writing in standard form, $4x^2 + 8x - 3 = 0$.

Comparing with $ax^2 + bx + c = 0$, evidently 4 corresponds to a, 8 to b, and -3 to c. Substituting these values in (F) gives

$$\begin{aligned} x &= \frac{-8 \pm \sqrt{64 - 4 \cdot 4 \cdot (-3)}}{2 \cdot 4} \\ &= \frac{-8 \pm \sqrt{64 + 48}}{8} = \frac{-8 \pm \sqrt{112}}{8} \\ &= \frac{-8 \pm 4\sqrt{7}}{8} = -1 \pm \frac{1}{2}\sqrt{7}. \end{aligned}$$

Check as usual.

2. $x^2 - 5x - 14 = 0.$

3. $x^2 - x - 1 = 0.$

4. $2x^2 + 5x + 2 = 0.$

5. $3x^2 + 5x = 4.$

6. $2x + 4 = x^2.$

7. $x^2 + x = 1.$

8. $x^2 - \frac{11x}{4} - \frac{15}{4} = 0.$

9. $6x + 7 = x^2.$

10. $21x = 1 - 72x^2.$

11. $2x^2 + 3x - 1 = 0.$

12. $5x = 3x^2 + 1.$

13. $x^2 + x\sqrt{5} = 10.$

14. $2k^2x^2 = kx + 1.$

Solution. Writing in standard form, $2k^2x^2 - kx - 1 = 0.$

Then $a = 2k^2$, $b = -k$, and $c = -1.$
Substituting these values in the formula (F),

$$x = \frac{-(-k) \pm \sqrt{(-k)^2 - 4 \cdot 2k^2(-1)}}{2 \cdot 2k^2}.$$

$$x = \frac{k \pm \sqrt{k^2 + 8k^2}}{4k^2} = \frac{k \pm 3k}{4k^2} = \frac{1}{k} \text{ and } -\frac{1}{2k}.$$

Check as usual.

15. $x^2 + 2kx - 3k^2 = 0.$

16. $x^2 - 2x\sqrt{a} - 3a = 0.$

17. $2a^2 = 9ax + 5x^2.$

18. $3x^2 + bx - 10b^2 = 0.$

19. $cx = 6x^2 - c^2.$

20. $a^2x^2 - 3abx + 2b^2 = 0.$

21. $p^2x^2 - 4pqx - 12q^2 = 0.$

22. $2n^2x^2 + 5nmx + 3m^2 = 0.$

23. $6h^2x^2 + 19hkx = 7k^2.$

24. $3a^2x^2 - 6abx + 2b^2 = 0.$

25. $x^2 + 2x = hx + 2h.$

Hint. $x^2 + (2 - h)x - 2h = 0.$ Then $a = 1$, $b = 2 - h$, and $c = -2h.$

Substituting these values in (F),

$$x = \frac{-(2-h) \pm \sqrt{(2-h)^2 - 4 \cdot 1 \cdot (-2h)}}{2}, \text{ etc.}$$

26. $x^2 + (p - q)x - pq = 0.$

27. $2x^2 + ab = 2bx + ax.$

28. $rsx^2 + sx = 3rx + 3.$

29. $mnx^2 + 4nx - 3mx = 12.$

30. $x^2 - 2hx = g - h^2.$

31. $kx^2 + kx - 2 = 2x.$

SUPPLEMENTARY EXERCISES INVOLVING EQUATIONS

Solve:

1. $\dfrac{x-3}{5}-\dfrac{x-5}{3}=2.$

2. $\dfrac{x+2}{9}-\dfrac{x-2}{4}=\dfrac{1}{6}.$

3. $\dfrac{2x-3}{3}-\dfrac{7-x}{4}=x-3.$

4. $\dfrac{x-6}{7}+\dfrac{x-4}{5}=\dfrac{2x+2}{35}.$

5. $\dfrac{x-2}{3}+\dfrac{x+5}{4}-\dfrac{x-1}{5}=\dfrac{29}{15}.$

6. $\dfrac{7}{x}+\dfrac{5}{x}=1.$

7. $\dfrac{3}{x}-\dfrac{4}{x}=\dfrac{1}{6}.$

8. $\dfrac{4}{x}-\dfrac{3}{4x}=\dfrac{13}{8}.$

9. $\dfrac{3}{2x}+\dfrac{1}{9}=\dfrac{5}{3x}.$

10. $\dfrac{x-6}{x}-\dfrac{3}{4}+\dfrac{x-3}{20}=0.$

11. $\frac{1}{2}(x+1)+\frac{1}{3}(x-\frac{7}{18})=0.$

12. $\dfrac{4x-3}{6}-\dfrac{3}{4}=\dfrac{7}{18}\left(\dfrac{4x+15}{14}\right).$

13. $\frac{1}{4}(2x^2-5x+3)=\frac{3}{8}(x^2-2x+1).$

14. $\dfrac{3x-4}{2}-\dfrac{x+5}{3}=\dfrac{1}{2}\cdot\dfrac{x-1}{4}.$

15. $\dfrac{x-4}{x}=\dfrac{x}{x+20}.$

16. $\dfrac{3x-4}{x+1}=\dfrac{6x-3}{2x+6}.$

17. $\dfrac{x-4}{x+1}=\dfrac{x-1}{x+9}.$

18. $\dfrac{5-x}{x+3}+\dfrac{x-1}{x+1}=\dfrac{2}{3}.$

19. $\dfrac{x+7}{2x-4}-\dfrac{x-5}{2x+1}=\dfrac{13}{4}.$

20. $\dfrac{x+3}{x+1}+\dfrac{x-4}{x+5}=\dfrac{2x-5}{x-3}.$

21. $\dfrac{3x-4}{2x+3}-\dfrac{2x-1}{4x+6}=-\dfrac{7}{6}.$

22. $\dfrac{x-4}{x+2}-\dfrac{x+3}{x-5}+\dfrac{14}{3}=0.$

23. $\dfrac{x+3}{x-3}+\dfrac{x-1}{x+4}=2.$

24. $\dfrac{x}{x^2-9} - \dfrac{x-1}{x+3} = \dfrac{8x-37}{8(3-x)}$.

25. $\dfrac{3(x+2)}{x^2-4} - \dfrac{1}{2-x} = \dfrac{6(x-3)}{2+x}$.

26. $\dfrac{x+\frac{1}{2}}{x-\frac{1}{2}} - \dfrac{x-\frac{1}{2}}{x+\frac{1}{2}} = x$.

27. $\dfrac{x^2-4x+3}{x^2-6x+8} = \dfrac{5}{8}$.

28. $\dfrac{x}{x^2-3x-4} = \dfrac{1}{4-x} - \dfrac{7}{1+x}$.

29. $.4x - 7.1 = 1.3x + .1$.

30. $.3(1.1x-5) = .03x - .6$.

31. $.14 - .9x = .15x$.

32. $.4(x-3) = .5(x+1)$.

33. $.2(4.1x - .7) = .7(.1x + .8)$.

34. $3.15x - 4.6 + x - 3.2 = .05x + .4$.

35. $.173x - 4.68 - .13x + 2.561 = .32x + .897 - .335x$.

36. $.65(.4x - .7) = .25(.9 - .32x)$.

37. $\dfrac{.4x-7}{6} = \dfrac{x-5.6}{5}$.

38. $\dfrac{.3x-.9}{.4} = \dfrac{4.4-6.3x}{3.2}$.

39. $\dfrac{.175x+.21}{.8} + \dfrac{3}{4}(.9x - .16) = .908x$.

40. $\dfrac{2.1-.8x}{.4x+.9} = \dfrac{.02x+.08}{.09+.01x}$.

41. $\dfrac{x-a}{b} = \dfrac{x-b}{a}$.

42. $\dfrac{a}{x} = \dfrac{x}{a}$.

43. $\dfrac{a^2}{x} = \dfrac{x}{b^2}$.

44. $\dfrac{x-a}{x-b} = \dfrac{a-b}{a+b}$.

45. $\dfrac{a-b}{x} = \dfrac{x}{a+b} - \dfrac{b^2}{ax+bx}$.

46. $ax = \dfrac{x}{a}$.

47. $\dfrac{x-a}{a} = \dfrac{x+b}{b}$.

48. $\dfrac{x}{a} + \dfrac{x}{b} = 5$.

49. $\dfrac{x}{a} - \dfrac{x}{b} = c$.

50. $\frac{a}{x} + \frac{b}{x} = 3.$

51. $\frac{a}{3x} - \frac{a}{5x} = \frac{1}{15}.$

52. $\frac{x-a}{b} = \frac{x+2b}{a} - 3.$

53. $\frac{a}{b} = \frac{x}{c-x}.$

54. $\frac{1}{x-a} + \frac{1}{x-b} = \frac{a+b}{(x-a)(x-b)}.$

55. $x - \frac{17a}{12c} + \frac{a^2}{2c^2x} = 0.$

56. A bubble of air of volume v units rising from a depth of d feet below the surface of the water expands to the volume V units at the top of the water according to the formula $V = \frac{d+34}{34}v$. Express v in terms of d and V. If a bubble starts from 100 feet below the surface and has a volume of .1 of a cubic inch at the top, find its volume when it started.

57. The formula $V = \frac{h(b+B+4m)}{6}$ gives the volume of a solid whose height, upper base, lower base, and mid-section are h, b, B, and m respectively. Find m in terms of the other letters.

58. Given the formula $s = vt + 16t^2$ for the distance (s) covered by a body projected downward with a velocity v in a time t, the units being feet and seconds, express v in terms of s and t.

59. Given $\frac{a}{b} - \frac{c}{d} = 0$. Express d in terms of the other letters.

60. Given $\frac{a-b}{b} = \frac{c-d}{d}$. Express d in terms of the other letters.

61. If $a = 2t$ and $b = 3t$, find the value of $\frac{\frac{a}{b} - \frac{b}{a}}{1 + \frac{b}{a}}$.

62. If $a = t$, $b = 2t$, $c = 3t$, find in terms of t:

$$\frac{1}{(a-b)(c-b)} + \frac{1}{(b-a)(c-a)} + \frac{1}{(a-c)(b-c)}.$$

63. Separate 60 into two parts such that $\frac{3}{5}$ the greater minus $\frac{1}{3}$ the less equals 22.

64. Separate 91 into two parts such that their quotient is $5\frac{1}{2}$.

65. Separate 70 into two parts such that 40 exceeds $\frac{3}{5}$ of the one by as much as the other exceeds 20.

66. The sum of two numbers is 13. Two thirds the greater plus $\frac{3}{5}$ the less equals 9. What are the numbers?

67. Two thirds a man's age now equals $\frac{3}{2}$ his age 25 years ago. What is his age?

68. The square of a certain number is 17 greater than $\frac{2}{3}$ of the product of the next two consecutive numbers. Find the number.

69. The length of a certain rectangle is $2\frac{1}{2}$ times the width. If it were a yard shorter and a yard and a half wider, its area would be 234 square feet greater. Find the dimensions of the rectangle.

70. A square court has the same area as a rectangular court whose length is 8 yards greater and whose width is 5 yards less. Find the dimensions of the square court.

71. A can do a piece of work in 2 days; B, in $2\frac{2}{3}$ days; C, in $3\frac{1}{5}$ days. How long will it take them, working together?

72. The difference between $\frac{3}{7}$ of a certain number and $\frac{1}{3}$ of it is 20. Find the number.

73. Four thousand dollars of Mr. A's income is not taxed. All of his income over that amount is taxed 2%, and all above ten thousand dollars is taxed 2% in addition. He pays a tax of $200. What is his income?

74. Given $t = \dfrac{3x^2 + x - 2}{3 - x}$, substitute this value for t in $\dfrac{3t + 9x + 2}{t + 3x + 2}$ and simplify.

75. If $\frac{1}{a}+\frac{1}{b}=1$, and if $\frac{a}{b}=c$, find b in terms of a alone, and a in terms of c alone.

76. Working 8 hours per day, A can do a certain piece of work in 7 days 4 hours, B in 3 days 6 hours. After a third man had done a quarter of the work, he struck, and A and B, taking his place, finished the work together. How long did it take them?

77. A and B start at the same time from two towns 100 miles apart and travel toward each other. Their respective rates are 8 and 12 miles per hour. Before they meet, A rests $2\frac{1}{2}$ hours and B rests 5 hours. How far does each go and how long is it before they meet?

78. A and B leave the same place at the same time for a point 90 miles distant, A traveling twice as fast as B. Upon reaching their destination A turns back and meets B 6 hours from the start. What are their rates?

79. A man rows upstream and back, a total distance of 30 miles, in 9 hours. His rate upstream was half his rate downstream. Find the rate of the current and his rate in still water.

80. A bolt of cloth is bought for \$324. Eight yards are cut off for use as samples, and the remainder sold at an advance of \$1 per yard, yielding a profit of \$76. Find the cost per yard.

81. Two automobiles each travel 72 miles, one going 4 miles per hour faster than the other and making the run in 15 minutes less time. Find the rate of each.

82. A and B start at the same time and place, but travel in opposite directions. B is delayed 2 hours on the way and travels 1 mile per hour slower than A. At the end of a certain time they are 172 miles apart. If A has then traveled 28 miles farther than B, find the number of hours since they started.

83. A and B together can do a piece of work in $3\frac{3}{5}$ days. B alone can do it in 3 days less than A. Find the number of days required by each.

84. At what time between 3 and 4 o'clock are the hands of the clock together?

HINT. Let $x =$ the number of minutes after 3. Then the hour hand covers $x - 15$ spaces and the minute hand x spaces, while the latter goes 12 times as fast as the former.

85. At what time between 7 and 8 are the hands of the clock together?

86. At what time between 9 and 10 are the hands of the clock together?

87. At what time between 1 and 2 are the hands of the clock in the same straight line but in opposite directions?

88. How long will it take to go m miles at d miles per hour?

89. What is the surface of a cubical box whose edge is d inches?

90. Express in dollars $p\%$ of x dollars.

91. If the average height of k boys is n inches, what is the sum of their heights in yards?

92. If $2n + 1$ bolts weigh x pounds, what is the weight of n of them?

93. To cook cereal in a fireless cooker, one uses $n + 2$ cups of water for n cups of dry cereal. How many cups of cereal may be used in a dish that holds k cups in all?

94. If it takes a man h hours to do a piece of work, what portion of the work can he do in 1 hour? What portion of the work would n men do in 1 hour? What portion would k men do in t hours?

95. If it takes m men b hours to do a piece of work, how long will it take r men to do it?

96. A man buys bananas at c cents a dozen and sells them for k cents each. What does he gain on d dozen?

97. A man buys oranges for d dollars per hundred and sells them k for a quarter. How much does he gain on h hundred?

98. A man bought x articles for c cents per hundred. He sold them all for $6. How many dollars did he lose?

99. A man buys goods for x dollars and sells them for $x - y$ dollars. What is his per cent of loss?

100. If y yards of ribbon cost d cents, find the cost of x yards.

101. If y yards of ribbon cost d cents, how many yards can be bought for k dollars?

102. A man buys goods for d dollars and sells them for h dollars. What is his per cent of gain?

103. One man can do a piece of work in d days, another can do the same work in f days. How many days will it take both, working together?

104. If it takes h hours to mow a acres, how many days of 10 hours each will it take to mow b acres?

105. A train goes f feet in t seconds. If this equals m miles per hour, write an equation involving f, t, and m.

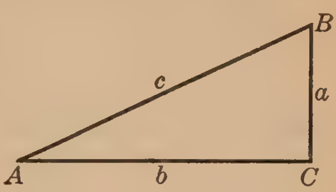
106. If n men can do a piece of work in d days, how many men would it be necessary to hire if the work had to be done in k days?

107. A transport plying between two ports is under fire for y yards of the way. If she steams k knots per hour, for how many minutes is she under fire? (1 knot = 6080 feet.)

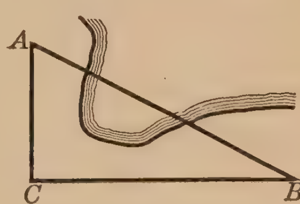
CHAPTER XXIII

INTRODUCTION TO TRIGONOMETRY

131. Introduction. The right triangle ABC has three angles and three sides which constitute the six *parts* of the triangle. In the sections that follow, methods will be explained for finding the remaining parts of any right triangle if the length of one side and the size of another part (other than the right angle) is known. That this problem is of great practical importance can be seen from the following illustration:



If A and B are two points near the shore of a lake, but separated by a bay, the distance between them cannot be measured directly. But if a point C can be found such that the angle ACB is a right angle, and if the line AC and the angle A can be measured, the methods that follow will enable us to find the length of AB. It is a simple matter for a surveyor to find the point C. Hence the distance AB can be ascertained.



132. Facts from geometry. Before going further it is necessary to state a few facts about the right triangle, some of which have already been used in arithmetic and all of which will be proved in geometry.

1. *A right angle contains 90 degrees.*

2. *The sum of the number of degrees in the angles of any triangle is 180 degrees.*

3. *The sum of the two acute angles of a right triangle is 90 degrees.*

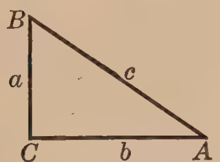


FIG. 1

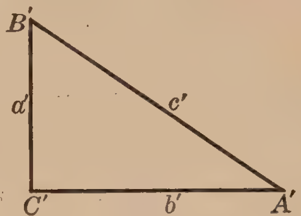


FIG. 2

4. *In a right triangle (Fig. 1), $c^2 = a^2 + b^2$ when a, b, and c denote the lengths of the sides and c is opposite the right angle.*

5. *If two triangles, ABC and $A'B'C'$ (Figs. 1 and 2), have their like lettered angles equal, then the ratios of like lettered sides are equal. That is,* $\frac{a}{a'} = \frac{c}{c'}$; $\frac{b}{b'} = \frac{c}{c'}$, *etc.*

The following queries refer to Fig. 1:

QUERY 1. If the angle $A = 70°$, how many degrees are there in $\angle B$? If angle $A = 27°$? If angle $A = 12°$?

QUERY 2. Can any angle of a right triangle contain more than 90°?

QUERY 3. If $a = 12$ inches and $b = 5$ inches, how long is c?

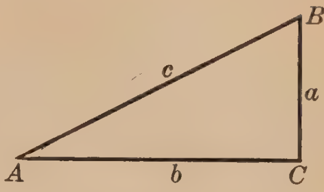
QUERY 4. If $b = 1$ foot and $c = 15$ inches, how long is a?

QUERY 5. It is necessary to know many distances that cannot be measured directly but must be determined somewhat as in the illustration of § 131. Give some examples of such distances.

133. Trigonometric ratios. In order to attack the problem stated in § 131 it is necessary to denote the ratios of the sides of a right triangle by certain names.

Let ABC be any right triangle with the right angle at C.

$$\text{The } \textit{sine} \text{ of the angle } A = \frac{a}{c} = \frac{\text{side opposite}}{\text{hypotenuse}}.$$



The *cosine* of the angle A

$$= \frac{b}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}}.$$

The *tangent* of the angle A

$$= \frac{a}{b} = \frac{\text{side opposite}}{\text{side adjacent}}.$$

These definitions are abbreviated as follows:

$$\sin A = \frac{a}{c}; \quad \cos A = \frac{b}{c}; \quad \tan A = \frac{a}{b}.$$

It is exceedingly important to note that the letters a, b, and c denote the *lengths* of the sides, all in terms of the same unit of measure. Thus, if $a = 6$ inches and $c = 2$ feet, sine $A = \frac{6}{24} = .25$.

QUERY 1. What is $\sin A$, $\cos A$, $\tan A$, if $a = 3$, $b = 4$, $c = 5$?

QUERY 2. Can $\sin A$ be greater than 1? If so, why? If not, why not?

QUERY 3. Can $\cos A$ be greater than 1? If so, why? If not, why not?

QUERY 4. Can $\tan A$ be greater than 1? If so, why? If not, why not?

QUERY 5. Give two different sets of values of a and b if $\tan A = 2$; if $\tan A = 5$.

QUERY 6. Give two different sets of values of a and c if $\sin A = \frac{2}{3}$; if $\sin A = \frac{4}{5}$.

134. Tables of trigonometric ratios. The values of the trigonometric ratios for all angles have been computed to many decimal places. The table on page 310 gives the values of the sine, the cosine, and the tangent correct to three places.

QUERY 1. From an inspection of the table would you say that $\sin A$ increases or decreases in value as angle A becomes larger?

Three-Place Trigonometric Table

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
1°	.017	.9998	.017	45°	.707	.707	1.000
2°	.035	.9994	.035	46°	.719	.695	1.036
3°	.052	.9986	.052	47°	.731	.682	1.072
4°	.070	.9976	.070	48°	.743	.669	1.111
5°	.087	.996	.087	49°	.755	.656	1.150
6°	.105	.995	.105	50°	.766	.643	1.192
7°	.122	.993	.123	51°	.777	.629	1.235
8°	.139	.990	.141	52°	.788	.616	1.280
9°	.156	.988	.158	53°	.799	.602	1.327
10°	.174	.985	.176	54°	.809	.588	1.376
11°	.191	.982	.194	55°	.819	.574	1.428
12°	.208	.978	.213	56°	.829	.559	1.483
13°	.225	.974	.231	57°	.839	.545	1.540
14°	.242	.970	.249	58°	.848	.530	1.600
15°	.259	.966	.268	59°	.857	.515	1.664
16°	.276	.961	.287	60°	.866	.500	1.732
17°	.292	.956	.306	61°	.875	.485	1.804
18°	.309	.951	.325	62°	.883	.469	1.881
19°	.326	.946	.344	63°	.891	.454	1.963
20°	.342	.940	.364	64°	.899	.438	2.050
21°	.358	.934	.384	65°	.906	.423	2.144
22°	.375	.927	.404	66°	.914	.407	2.246
23°	.391	.921	.424	67°	.921	.391	2.356
24°	.407	.914	.445	68°	.927	.375	2.475
25°	.423	.906	.466	69°	.934	.358	2.605
26°	.438	.899	.488	70°	.940	.342	2.747
27°	.454	.891	.510	71°	.946	.326	2.904
28°	.469	.883	.532	72°	.951	.309	3.078
29°	.485	.875	.554	73°	.956	.292	3.271
30°	.500	.866	.577	74°	.961	.276	3.487
31°	.515	.857	.601	75°	.966	.259	3.732
32°	.530	.848	.625	76°	.970	.242	4.011
33°	.545	.839	.649	77°	.974	.225	4.331
34°	.559	.829	.675	78°	.978	.208	4.705
35°	.574	.819	.700	79°	.982	.191	5.145
36°	.588	.809	.727	80°	.985	.174	5.671
37°	.602	.799	.754	81°	.988	.156	6.314
38°	.616	.788	.781	82°	.990	.139	7.115
39°	.629	.777	.810	83°	.993	.122	8.144
40°	.643	.766	.839	84°	.995	.105	9.514
41°	.656	.755	.869	85°	.996	.087	11.43
42°	.669	.743	.900	86°	.9976	.070	14.30
43°	.682	.731	.933	87°	.9986	.052	19.08
44°	.695	.719	.966	88°	.9994	.035	28.64
45°	.707	.707	1.000	89°	.9998	.017	57.29

QUERY 2. Would you say that $\cos A$ increases or decreases as A becomes larger?

QUERY 3. Would you say that tangent A increases or decreases as A becomes larger?

QUERY 4. Read from the table

$\sin 4°$; $\sin 28°$; $\sin 47°$; $\sin 76°$.
$\cos 89°$; $\cos 35°$; $\cos 60°$; $\cos 8°$.
$\tan 45°$; $\tan 71°$; $\tan 5°$; $\tan 86°$.

QUERY 5. From the table determine the number of degrees in angle A if

(a) $\sin A = .438$	(d) $\cos A = .966$	(g) $\tan A = 1.732$
(b) $\sin A = .982$	(e) $\cos A = .225$	(h) $\tan A = .445$
(c) $\sin A = .707$	(f) $\cos A = .755$	(i) $\tan A = 4.011$

QUERY 6. From the table determine the number of degrees of the angle that most nearly satisfies the following:

(a) $\sin A = .841$	(d) $\cos A = .983$	(g) $\tan A = .262$
(b) $\sin A = .969$	(e) $\cos A = .425$	(h) $\tan A = .799$
(c) $\sin A = .180$	(f) $\cos A = .815$	(i) $\tan A = 3.200$

QUERY 7.

$\sin 30° = ?$ $\cos 60° = ?$
$\sin 47° = ?$ $\cos 43° = ?$
$\sin 20° = ?$ $\cos (90° - 20°) = ?$
$\cos 38° = ?$ $\cos (90° - 38°) = ?$

QUERY 8. If $A + B = 90°$, what relation exists between $\sin A$ and $\cos B$? $\cos A$ and $\sin B$?

QUERY 9. From $\sin A = \frac{a}{c}$ express a in terms of $\sin A$ and c.

QUERY 10. From $\cos A = \frac{b}{c}$ find b; find c.

QUERY 11. Solve $\tan A = \frac{a}{b}$ for a; for b.

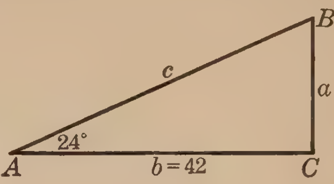
QUERY 12. Solve $\cos 27° = \frac{b}{90}$ for b.

HINT. From the table $\cos 27° = .891$.

QUERY 13. Solve $\tan 40° = \frac{12}{b}$ for b.

QUERY 14. Solve $\sin 63° = \frac{42}{c}$ for c.

135. Use of trigonometric ratios. If one side and one angle of a right triangle can in any way be determined, the other parts of the triangle may be computed by means of the trigonometric ratios.



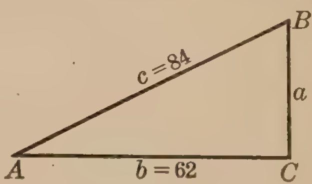
EXAMPLE 1. In the triangle ABC the angle $A = 24°$ and $b = 42$. Find BC.

Solution.
$$\tan A = \frac{a}{b}.$$
$$\tan 24° = \frac{a}{42}.$$
$$a = 42 \times \tan 24°$$
$$= 42\,(.445) = 18.69.$$
Then a, or BC, $= 18.69$.

From the definitions of the trigonometric ratios in §133 it appears that each definition contains three elements, two sides and one angle. Hence, if any two of these are known, the third can be found.

For example, if a and c are known, $\sin A$ can be found, and from the table the value of A can be determined approximately.

Since the sum of the acute angles of a right triangle is 90°, if one is known the other can be found immediately.



EXAMPLE 2. In the triangle ABC, $b = 62$, $c = 84$. Find A.

Solution.
$$\cos A = \frac{b}{c}.$$
$$\cos A = \frac{62}{84} = .738.$$
$$A = 42° \text{ approximately.}$$

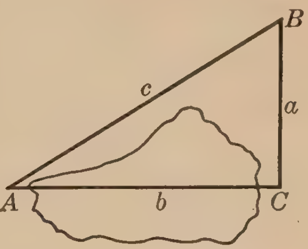
EXERCISES

1. Given $A = 25°$, $c = 28$. Find B, a, and b.
2. Given $A = 69°$, $b = 57$. Find B, a, and c.
3. Given $A = 47°$, $a = 21$. Find B, b, and c.
4. Given $B = 73°$, $c = 51$. Find A, a, and b.
5. Given $B = 31°$, $b = 100$. Find A, a, and c.
6. Given $B = 8°$, $a = 84$. Find A, b, and c.
7. Given $A = 27°$, $b = 48$. Find a, B, and c.
8. Given $A = 84°$, $c = 90$. Find a, B, and b.
9. Given $B = 65°$, $c = 147$. Find A, b, and a.

10. The shadow of a tree on level ground is 60 feet long and the sun's rays make an angle of 52° with the ground. How high is the tree?

11. A ladder 50 feet long rests against the side of a house and reaches a point on the house 30 feet from the ground. What angle does it make with the ground?

12. If the edge of the Great Pyramid is 609 feet long and it makes an angle of 52° with the horizontal, what is the height of the pyramid?



13. A surveyor finds that the angle B in the adjacent figure is 54°, $C = 90°$, and the line AB is found to be 627 feet long. How long is the line AC?

14. A ship sails southwest a distance of 36 miles. How far is it south of the starting-point?

15. A flagstaff 85 feet high casts a shadow 123 feet long. What is the angle of elevation of the sun?

16. A ladder rests against the side of a building, making an angle of 30° with the ground. The foot of the ladder is 18 feet from the building. How long is the ladder?

136. Four-place tables of trigonometric ratios. For many purposes more accurate numerical results are desired than it is possible to obtain by the use of three-place tables. If the measurement of lines is made with an accuracy that ensures only three correct places, no more accurate tables are needed. For numerical results cannot be attained with any greater degree of accuracy than the least accurate of the measurements with which one starts. It is possible, however, both in the laboratory and in the field to take measurements so accurately that four-place tables are desirable.

Such tables are found on pages 317–319. These tables give the three ratios already defined for every ten minutes of angle.

By the process of interpolation it is possible to find the values of these ratios exact to the nearest minute.

Interpolation depends on the assumption that the change in the ratio is proportional to the change in the corresponding angle for the 10-minute intervals.

There are two problems of interpolation:

(a) Given an angle not in the table to find the corresponding sine, cosine, or tangent.

(b) Given the value of a sine, a cosine, or a tangent not in the table to find the corresponding angle.

EXAMPLE 1. What are the values of $\sin 34° 25'$ and of $\sin 34° 28'$?

Solution. From the table we find that $\sin 34° 20' = .5640$ and $\sin 34° 30' = .5664$. We assume that $\sin 34° 25'$ lies just halfway between .5640 and .5664, or at .5652. Sin $34° 28'$ would be .8 of the way from .5640 to .5664, or at .5659.

A similar procedure should be followed in finding the tangent or the cosine of an angle correct for minutes.

When the numerical value of the ratio is given and it is desired to find the angle for the nearest minute, the following operation is employed.

EXAMPLE 2. It is desired to find A when $\sin A = .4423$.

Solution. Reference to the tables shows that since .4423 lies between .4410 and .4436, A must lie between 26° 10′ and 26° 20′. The entire difference between .4410 and .4436, which is 26 in the last two places, corresponds to the entire 10′ between 26° 10′ and 26° 20′. But .4423 is 13 in the last two places greater than .4410. Hence this 13 corresponds to $\frac{13}{26}$, or $\frac{1}{2}$, of the whole 10′ between 26° 10′ and 26° 20′. That is, $A = 26° 15'$.

This work may be abbreviated as follows:

Given $\tan A = 1.2443$. Find A.

$$\begin{array}{ll} \tan 51° 20' = 1.2497\,; & \tan A \qquad\quad = 1.2443 \\ \tan 51° 10' = \underline{1.2423}\,; & \tan 51° 10' = \underline{1.2423} \\ \qquad\qquad\quad 74 & \qquad\qquad\qquad 20 \end{array}$$

$$\tfrac{20}{74} \text{ of } 10' = 3'.$$

Hence $$A = 51° 13'.$$

In interpolating with the cosine ratio it is necessary to remember that as the angle increases the cosine decreases.

EXERCISES

Find from the tables on pages 317–319:

1. $\sin 18° 40'$.
2. $\cos 36° 20'$.
3. $\tan 40° 10'$.
4. $\cos 60° 30'$.
5. $\tan 70° 50'$.
6. $\sin 80° 20'$.

Verify the following statements:

7. $\sin 35° 25' = .5795$.
8. $\sin 53° 5' = .7995$.
9. $\tan 25° 42' = .4813$.
10. $\tan 41° 38' = .8889$.
11. $\cos 63° 17' = .4496$.
12. $\cos 41° 33' = .7484$.
13. $\tan 61° 34' = 1.8469$.
14. $\cos 15° 2' = .9658$.

137. The cotangent. It is often a convenience in practical work to have a fourth ratio, called the *cotangent.* With reference to the figure of § 133,

$$\cot A = \frac{b}{a} = \frac{\text{side adjacent}}{\text{side opposite}}.$$

138. Explanation of the arrangement of the tables. From the definitions of § 133 it is clear that sin 31° equals cos 59° and sin 62° equals cos 28°. Further, if two sides of a right triangle are equal, the angles opposite are equal and each is 45°. Hence sin 45° equals cos 45°. It follows from these simple relations that the sine of any angle between 0° and 45° equals the cosine of an angle between 45° and 90°, and the sine of an angle between 45° and 90° equals the cosine of an angle between 0° and 45°. Advantage is taken of this relation in the arrangement of the tables, and the number of pages necessary is half what it would otherwise be.

Referring to the tables on pages 317–319, the sines of angles less than 45° are obtained by noting the given angle in the left-hand column, and the required ratio is the number opposite the angle in the column headed *sine.* Similarly for the cosine.

If an angle is between 45° and 90°, the angle is read from the right-hand column and the sine is the number opposite, in the column at the *foot* of which is the word *sine.* Similarly for the cosine.

For the tangent and the cotangent a similar method holds. This follows from their definition. A brief inspection of the tables will verify this statement.

Find by interpolation:

1. sin 12° 18′.	**4.** sin 75° 12′.	**7.** cos 18° 18′.
2. tan 35° 42′.	**5.** cos 18° 15′.	**8.** cos 24° 48′.
3. tan 80° 36′.	**6.** cos 18° 12′.	**9.** cos 72° 56′.

°	SIN	COS	TAN	COT		°	SIN	COS	TAN	COT	
0°	.0000	1.0000	.0000	∞	**90°**	**8°**	.1392	.9903	.1405	7.1154	**82°**
10′	.0029	1.0000	.0029	343.77	50′	10′	.1421	.9899	.1435	6.9682	50′
20′	.0058	1.0000	.0058	171.89	40′	20′	.1449	.9894	.1465	6.8269	40′
30′	.0087	1.0000	.0087	114.59	30′	30′	.1478	.9890	.1495	6.6912	30′
40′	.0116	.9999	.0116	85.940	20′	40′	.1507	.9886	.1524	6.5606	20′
50′	.0145	.9999	.0145	68.750	10′	50′	.1536	.9881	.1554	6.4348	10′
1°	.0175	.9998	.0175	57.290	**89°**	**9°**	.1564	.9877	.1584	6.3138	**81°**
10′	.0204	.9998	.0204	49.104	50′	10′	.1593	.9872	.1614	6.1970	50′
20′	.0233	.9997	.0233	42.964	40′	20′	.1622	.9868	.1644	6.0844	40′
30′	.0262	.9997	.0262	38.188	30′	30′	.1650	.9863	.1673	5.9758	30′
40′	.0291	.9996	.0291	34.368	20′	40′	.1679	.9858	.1703	5.8708	20′
50′	.0320	.9995	.0320	31.242	10′	50′	.1708	.9853	.1733	5.7694	10′
2°	.0349	.9994	.0349	28.636	**88°**	**10°**	.1736	.9848	.1763	5.6713	**80°**
10′	.0378	.9993	.0378	26.432	50′	10′	.1765	.9843	.1793	5.5764	50′
20′	.0407	.9992	.0407	24.542	40′	20′	.1794	.9838	.1823	5.4845	40′
30′	.0436	.9990	.0437	22.904	30′	30′	.1822	.9833	.1853	5.3955	30′
40′	.0465	.9989	.0466	21.470	20′	40′	.1851	.9827	.1883	5.3093	20′
50′	.0494	.9988	.0495	20.206	10′	50′	.1880	.9822	.1914	5.2257	10′
3°	.0523	.9986	.0524	19.081	**87°**	**11°**	.1908	.9816	.1944	5.1446	**79°**
10′	.0552	.9985	.0553	18.075	50′	10′	.1937	.9811	.1974	5.0658	50′
20′	.0581	.9983	.0582	17.169	40′	20′	.1965	.9805	.2004	4.9894	40′
30′	.0610	.9981	.0612	16.350	30′	30′	.1994	.9799	.2035	4.9152	30′
40′	.0640	.9980	.0641	15.605	20′	40′	.2022	.9793	.2065	4.8430	20′
50′	.0669	.9978	.0670	14.924	10′	50′	.2051	.9787	.2095	4.7729	10′
4°	.0698	.9976	.0699	14.301	**86°**	**12°**	.2079	.9781	.2126	4.7046	**78°**
10′	.0727	.9974	.0729	13.727	50′	10′	.2108	.9775	.2156	4.6382	50′
20′	.0756	.9971	.0758	13.197	40′	20′	.2136	.9769	.2186	4.5736	40′
30′	.0785	.9969	.0787	12.706	30′	30′	.2164	.9763	.2217	4.5107	30′
40′	.0814	.9967	.0816	12.251	20′	40′	.2193	.9757	.2247	4.4494	20′
50′	.0843	.9964	.0846	11.826	10′	50′	.2221	.9750	.2278	4.3897	10′
5°	.0872	.9962	.0875	11.430	**85°**	**13°**	.2250	.9744	.2309	4.3315	**77°**
10′	.0901	.9959	.0904	11.059	50′	10′	.2278	.9737	.2339	4.2747	50′
20′	.0929	.9957	.0934	10.712	40′	20′	.2306	.9730	.2370	4.2193	40′
30′	.0958	.9954	.0963	10.385	30′	30′	.2334	.9724	.2401	4.1653	30′
40′	.0987	.9951	.0992	10.078	20′	40′	.2363	.9717	.2432	4.1126	20′
50′	.1016	.9948	.1022	9.7882	10′	50′	.2391	.9710	.2462	4.0611	10′
6°	.1045	.9945	.1051	9.5144	**84°**	**14°**	.2419	.9703	.2493	4.0108	**76°**
10′	.1074	.9942	.1080	9.2553	50′	10′	.2447	.9696	.2524	3.9617	50′
20′	.1103	.9939	.1110	9.0098	40′	20′	.2476	.9689	.2555	3.9136	40′
30′	.1132	.9936	.1139	8.7769	30′	30′	.2504	.9681	.2586	3.8667	30′
40′	.1161	.9932	.1169	8.5555	20′	40′	.2532	.9674	.2617	3.8208	20′
50′	.1190	.9929	.1198	8.3450	10′	50′	.2560	.9667	.2648	3.7760	10′
7°	.1219	.9925	.1228	8.1443	**83°**	**15°**	.2588	.9659	.2679	3.7321	**75°**
10′	.1248	.9922	.1257	7.9530	50′	10′	.2616	.9652	.2711	3.6891	50′
20′	.1276	.9918	.1287	7.7704	40′	20′	.2644	.9644	.2742	3.6470	40′
30′	.1305	.9914	.1317	7.5958	30′	30′	.2672	.9636	.2773	3.6059	30′
40′	.1334	.9911	.1346	7.4287	20′	40′	.2700	.9628	.2805	3.5656	20′
50′	.1363	.9907	.1376	7.2687	10′	50′	.2728	.9621	.2836	3.5261	10′
	COS	SIN	COT	TAN	°		COS	SIN	COT	TAN	°

°	SIN	COS	TAN	COT		°	SIN	COS	TAN	COT	
16°	.2756	.9613	.2867	3.4874	**74°**	**24°**	.4067	.9135	.4452	2.2460	**66°**
10′	.2784	.9605	.2899	3.4495	50′	10′	.4094	.9124	.4487	2.2286	50′
20′	.2812	.9596	.2931	3.4124	40′	20′	.4120	.9112	.4522	2.2113	40′
30′	.2840	.9588	.2962	3.3759	30′	30′	.4147	.9100	.4557	2.1943	30′
40′	.2868	.9580	.2994	3.3402	20′	40′	.4173	.9088	.4592	2.1775	20′
50′	.2896	.9572	.3026	3.3052	10′	50′	.4100	.9075	.4628	2.1609	10′
17°	.2924	.9563	.3057	3.2709	**73°**	**25°**	.4226	.9063	.4663	2.1445	**65°**
10′	.2952	.9555	.3089	3.2371	50′	10′	.4253	.9051	.4699	2.1283	50′
20′	.2979	.9546	.3121	3.2041	40′	20′	.4279	.9038	.4734	2.1123	40′
30′	.3007	.9537	.3153	3.1716	30′	30′	.4305	.9026	.4770	2.0965	30′
40′	.3035	.9528	.3185	3.1397	20′	40′	.4331	.9013	.4806	2.0809	20′
50′	.3062	.9520	.3217	3.1084	10′	50′	.4358	.9001	.4841	2.0655	10′
18°	.3090	.9511	.3249	3.0777	**72°**	**26°**	.4384	.8988	.4877	2.0503	**64°**
10′	.3118	.9502	.3281	3.0475	50′	10′	.4410	.8975	.4913	2.0353	50′
20′	.3145	.9492	.3314	3.0178	40′	20′	.4436	.8962	.4950	2.0204	40′
30′	.3173	.9483	.3346	2.9887	30′	30′	.4462	.8949	.4986	2.0057	30′
40′	.3201	.9474	.3378	2.9600	20′	40′	.4488	.8936	.5022	1.9912	20′
50′	.3228	.9465	.3411	2.9319	10′	10′	.4514	.8923	.5059	1.9768	10′
19°	.3256	.9455	.3443	2.9042	**71°**	**27°**	.4540	.8910	.5095	1.9626	**63°**
10′	.3283	.9446	.3476	2.8770	50′	10′	.4566	.8897	.5132	1.9486	50′
20′	.3311	.9436	.3408	2.8502	40′	20′	.4592	.8884	.5169	1.9347	40′
30′	.3338	.9426	.3541	2.8239	30′	30′	.4617	.8870	.5206	1.9210	30′
40′	.3365	.9417	.3574	2.7980	20′	40′	.4643	.8857	.5243	1.9074	20′
50′	.3393	.9407	.3607	2.7725	10′	50′	.4669	.8843	.5280	1.8940	10′
20°	.3420	.9397	.3640	2.7475	**70°**	**28°**	.4695	.8829	.5317	1.8807	**62°**
10′	.3448	.9387	.3673	2.7228	50′	10′	.4720	.8816	.5354	1.8676	50′
20′	.3475	.9377	.3706	2.6985	40′	20′	.4746	.8802	.5392	1.8546	40′
30′	.3502	.9367	.3739	2.6746	30′	30′	.4772	.8788	.5430	1.8418	30′
40′	.3529	.9356	.3772	2.6511	20′	40′	.4797	.8774	.5467	1.8291	20′
50′	.3557	.9346	.3805	2.6279	10′	50′	.4823	.8760	.5505	1.8165	10′
21°	.3584	.9336	.3839	2.6051	**69°**	**29°**	.4848	.8746	.5543	1.8040	**61°**
10′	.3611	.9325	.3872	2.5826	50′	10′	.4874	.8732	.5581	1.7917	50′
20′	.3638	.9315	.3906	2.5605	40′	20′	.4899	.8718	.5619	1.7796	40′
30′	.3665	.9304	.3939	2.5386	30′	30′	.4924	.8704	.5658	1.7675	30′
40′	.3692	.9293	.3973	2.5172	20′	40′	.4950	.8689	.5696	1.7556	20′
50′	.3719	.9283	.4006	2.4960	10′	50′	.4975	.8675	.5735	1.7437	10′
22°	.3746	.9272	.4040	2.4751	**68°**	**30°**	.5000	.8660	.5774	1.7321	**60°**
10′	.3773	.9261	.4074	2.4545	50′	10′	.5025	.8646	.5812	1.7205	50′
20′	.3800	.9250	.4108	2.4342	40′	20′	.5050	.8631	.5851	1.7090	40′
30′	.3827	.9239	.4142	2.4142	30′	30′	.5075	.8616	.5890	1.6977	30′
40′	.3854	.9228	.4176	2.3945	20′	40′	.5100	.8601	.5930	1.6864	20′
50′	.3881	.9216	.4210	2.3750	10′	50′	.5125	.8587	.5969	1.6753	10′
23°	.3907	.9205	.4245	2.3559	**67°**	**31°**	.5150	.8572	.6009	1.6643	**59°**
10′	.3934	.9194	.4279	2.3369	50′	10′	.5175	.8557	.6048	1.6534	50′
20′	.3961	.9182	.4314	2.3183	40′	20′	.5200	.8542	.6088	1.6426	40′
30′	.3987	.9171	.4348	2.2998	30′	30′	.5225	.8526	.6128	1.6319	30′
40′	.4014	.9159	.4383	2.2817	20′	40′	.5250	.8511	.6168	1.6212	20′
50′	.4041	.9147	.4417	2.2637	10′	50′	.5275	.8496	.6208	1.6107	10′
	COS	SIN	COT	TAN	°		COS	SIN	COT	TAN	°

°	SIN	COS	TAN	COT	
32°	.5299	.8480	.6249	1.6003	**58°**
10′	.5324	.8465	.6289	1.5900	50′
20′	.5348	.8450	.6330	1.5798	40′
30′	.5373	.8434	.6371	1.5697	30′
40′	.5398	.8418	.6412	1.5597	20′
50′	.5422	.8403	.6453	1.5497	10′
33°	.5446	.8387	.6494	1.5399	**57°**
10′	.5471	.8371	.6536	1.5301	50′
20′	.5495	.8355	.6577	1.5204	40′
30′	.5519	.8339	.6619	1.5108	30′
40′	.5544	.8323	.6661	1.5013	20′
50′	.5568	.8307	.6703	1.4919	10′
34°	.5592	.8290	.6745	1.4826	**56°**
10′	.5616	.8274	.6787	1.4733	50′
20′	.5640	.8258	.6830	1.4641	40′
30′	.5664	.8241	.6873	1.4550	30′
40′	.5688	.8225	.6916	1.4460	20′
50′	.5712	.8208	.6959	1.4370	10′
35°	.5736	.8192	.7002	1.4281	**55°**
10′	.5760	.8175	.7046	1.4193	50′
20′	.5783	.8158	.7089	1.4106	40′
30′	.5807	.8141	.7133	1.4019	30′
40′	.5831	.8124	.7177	1.3934	20′
50′	.5854	.8107	.7221	1.3848	10′
36°	.5878	.8090	.7265	1.3764	**54°**
10′	.5901	.8073	.7310	1.3680	50′
20′	.5925	.8056	.7355	1.3597	40′
30′	.5948	.8039	.7400	1.3514	30′
40′	.5972	.8021	.7445	1.3432	20′
50′	.5995	.8004	.7490	1.3351	10′
37°	.6018	.7986	.7536	1.3270	**53°**
10′	.6041	.7969	.7581	1.3190	50′
20′	.6065	.7951	.7627	1.3111	40′
30′	.6088	.7934	.7673	1.3032	30′
40′	.6111	.7916	.7720	1.2954	20′
50′	.6134	.7898	.7766	1.2876	10′
38°	.6157	.7880	.7813	1.2799	**52°**
10′	.6180	.7862	.7860	1.2723	50′
20′	.6202	.7844	.7907	1.2647	40′
30′	.6225	.7826	.7954	1.2572	30′
40′	.6248	.7808	.8002	1.2497	20′
50′	.6271	.7790	.8050	1.2423	10′
39°	.6293	.7771	.8098	1.2349	**51°**
10′	.6316	.7753	.8146	1.2276	50′
20′	.6338	.7735	.8195	1.2203	40′
30′	.6361	.7716	.8243	1.2131	30′
40′	.6383	.7698	.8292	1.2059	20′
50′	.6406	.7679	.8342	1.1988	10′
	COS	SIN	COT	TAN	°

°	SIN	COS	TAN	COT	
40°	.6428	.7660	.8391	1.1918	**50°**
10′	.6450	.7642	.8441	1.1847	50′
20′	.6472	.7623	.8491	1.1778	40′
30′	.6494	.7604	.8541	1.1708	30′
40′	.6517	.7585	.8591	1.1640	20′
50′	.6539	.7566	.8642	1.1571	10′
41°	.6561	.7547	.8693	1.1504	**49°**
10′	.6583	.7528	.8744	1.1436	50′
20′	.6604	.7509	.8796	1.1369	40′
30′	.6626	.7490	.8847	1.1303	30′
40′	.6648	.7470	.8899	1.1237	20′
50′	.6670	.7451	.8952	1.1171	10′
42°	.6691	.7431	.9004	1.1106	**48°**
10′	.6713	.7412	.9057	1.1041	50′
20′	.6734	.7392	.9110	1.0977	40′
30′	.6756	.7373	.9163	1.0913	30′
40′	.6777	.7353	.9217	1.0850	20′
50′	.6799	.7333	.9271	1.0786	10′
43°	.6820	.7314	.9325	1.0724	**47°**
10′	.6841	.7294	.9380	1.0661	50′
20′	.6862	.7274	.9435	1.0599	40′
30′	.6884	.7254	.9490	1.0538	30′
40′	.6905	.7234	.9545	1.0477	20′
50′	.6926	.7214	.9601	1.0416	10′
44°	.6947	.7193	.9657	1.0355	**46°**
10′	.6967	.7173	.9713	1.0295	50′
20′	.6988	.7153	.9770	1.0235	40′
30′	.7009	.7133	.9827	1.0176	30′
40′	.7030	.7112	.9884	1.0117	20′
50′	.7050	.7092	.9942	1.0058	10′
45°	.7071	.7071	1.0000	1.0000	**45°**
	COS	SIN	COT	TAN	°

In Exercises 11–28 the triangles are understood to be right triangles.

10. Given $A = 40° 15'$, $c = 100$. Find a.

11. Given $B = 32° 24'$, $a = 60$. Find b.

12. Given $A = 64° 18'$, $c = 28$. Find a.

13. Given $B = 54° 32'$, $b = 40$. Find c.

14. Given $a = 32$, $c = 51$. Find A.

15. Given $b = 45$, $c = 80$. Find B.

16. Given $a = 110$, $b = 4$. Find A.

17. Given $a = 3$, $b = 4$. Find the angles and the hypotenuse.

18. Given $c = 26$, $a = 10$. Find the other parts.

Solve the following right triangles:

19. $A = 30°$, $c = 1325$.

20. $B = 45°$, $a = 856$.

21. $c = 63.7$, $A = 27° 15'$.

22. $C = 3.428$, $B = 50° 44'$.

23. $a = .0725$, $A = 72° 52'$.

24. $b = 1888$, $A = 41° 37'$.

25. $b = 318$, $a = 250$.

26. $c = 423$, $b = 184$.

27. $c = 1296$, $a = 1024$.

28. The Eiffel Tower is about 986 feet high. At a certain time it casts a shadow 1200 feet long. What angle do the sun's rays then make with the ground?

29. A boat travels southwest 318.4 miles. It is then how many miles south of its starting-point?

30. A balloon is anchored by a rope 875 feet long. When the air is quiet a man in the balloon observes that the line from his eye to an object on the ground makes an angle of $30° 20'$ with the horizontal. How far is he from the object?

31. Two men on a level field observed a balloon and found its direction with respect to the horizontal to be 40° and 62° respectively. The balloon was 4000 feet high and in the same vertical plane with the two men. How far were they apart?

32. The sides a and b of a triangle (not a right triangle) are 200 feet and 156 feet respectively. Their included angle C is 50°. A line is drawn from A, making a right angle with the opposite side. This line is an altitude of the triangle. Find it. Find the area of the triangle.

33. Two sides of a triangle are 120 and 144 respectively, and their included angle is 75°. Find one altitude and the area of the triangle.

34. From two points A and B on shore 1000 feet apart a boat P is observed. The angle $BAP = 32°$ and the angle $ABP = 48°$. Find the length of the perpendicular from P to the line AB.

HINT. Draw the triangle ABP and the perpendicular PK. Let $AK = x$. Then $BK = 1000 - x$ and $\tan 32° = \frac{PK}{x}$ (1), and $\tan 48° = \frac{PK}{1000 - x}$ (2). Solve (1) and (2).

35. Find AP and BP in Ex. 34.

36. Assuming that the Mt. Washington railway rises 3596 feet in traveling along 3 miles of track, what average angle does the track make with the horizontal?

37. A tree 75 feet high stands on the bank of a river, and subtends an angle of 32° 42′ on the opposite bank. What is the width of the river?

38. A guy-rope from the top of a vertical 120-foot pole is attached to the ground 45 feet from the base of the pole. What angle does the rope make with the ground, assuming the ground to be horizontal?

39. In certain states the highway commissions will not allow public roads to be built with a gradient steeper than 1 foot of rise for every 10 feet of roadway. What angle does such a road make with the horizontal?

40. An airplane is above a point 2 miles from an observer. The line of sight from the observer to the airplane makes an angle of $25° 42'$ with the horizontal. How high is the airplane above the ground?

41. A railway embankment is 8 feet high. Its width at the top is 10 feet and at the bottom is 25 feet. What angle do the sides make with the horizontal?

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